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STATISTICAL MECHANICS OF NONLINEAR NONEQUILIBRIUM FINANCIAL MARKETS

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Abstract—An approach to understanding the nature of markets is modelled using methods of modern nonlinear nonequilibrium statistical mechanics. This permits examination of the premise that markets can be described by nonlinear nonequilibrium Markovian distributions. Corrections to previous nonlinear continuous time models are explicitly presented. A quite general microscopic model is presented of individual agents operating on a market, and explicit relationships are derived between variables describing these agents and the macroscopic market.

1. INTRODUCTION

Several studies imply that changing prices of many markets do not follow a random walk, that they may have long-term dependences in price correlations, and that they may not be efficient in quickly arbitraging new information [1-5]. However, these and other studies have at least several shortcomings, some of which are pointed out by their authors:

(A) A random walk for returns, rate of change of prices over prices, is described by a Langevin equation with simple additive noise η, typically representing the continual random influx of information into the market.

$$\dot{\Gamma} = -\gamma_1 + \gamma_2 \eta \ , \tag{1}$$

$$\dot{\Gamma} = \mathrm{d}\Gamma/\mathrm{d}t \ ,$$

$$< \eta(t) >_\eta = 0 \ ,$$

$$< \eta(t), \eta(t') >_\eta = \delta(t - t') \ ,$$

where γ_1 and γ_2 are constants, and Γ is the logarithm of (scaled) price. From this equation, other models may be derived, such as the times-series model and the Kalman filter method of control theory [6]. However, in the process of this transformation, the Markovian description typically is lost by projection onto a smaller state space [7]. In this context, price, although the most dramatic observable, may not be the only appropriate dependent variable or order parameter for the system of markets [8]. This possibility has also been called the ‘‘semistrong form of the efficient market hypothesis’’ [3].

This paper only considers Gaussian noise, ‘‘white’’ or ‘‘colored’’ (e.g., γ_2 not constant, also called ‘‘multiplicative’’ noise). These methods are not conveniently used for other sources of noise also currently considered by economists, e.g., Poisson processes [9] or Bernoulli processes [10,11]. It remains to be seen if colored noise can emulate these processes in the empirical ranges of interest, in some reasonable limits [12]. For example, within limited ranges, log-normal distributions can approximate $1/f$ distributions, and Pareto-Lévy tails may be modelled as subordinated log-normal distribtutions with amplification mechanisms [13].

(B) It is also necessary to explore the possibilities that a given market evolves in nonequilibrium, e.g., evolving irreversibly, as well as nonlinearly, e.g., $\gamma_{1,2}$ may be functions of Γ. Irreversibility, e.g., causality [14] and nonlinearity [15], have been suggested as processes necessary to take into account in order to understand markets, but modern methods of statistical mechanics now provide a more explicit paradigm to consistently include these processes in *bona fide* probability distributions. Reservations have been expressed about these earlier models at the time of their presentation [16]. It should also be noted that considerations of general Martingale processes, which conclude that markets behave nonrandomly, typically have been restricted to additive random processes [4].

(C) Besides assuming a rather specialized form for a Markovian process, Eq. (1) also assumes that real time is the proper independent variable. This is true for physical and most biophysical processes that have relatively continuous interactions [17-20], but for social and economic systems, some other density of relevant events might better describe the temporal evolution of the system.

For example, typically, t is measured by a small time unit $\hat{t}$ that averages over a chosen number of ticks/trades and $\bar{t}$ is a macroscopic epoch. Reasonable values of $\hat{t}$ and $\bar{t}$ are on the order of minutes and days, respectively. A mesoscopic time scale τ, $\bar{t} > \tau > \hat{t}$, and a ‘‘smoothness’’ parameter Γ_γ, a fraction of $\Gamma(\bar{t})$, are chosen to search and fit $\Gamma(t)$ to local minima and maxima. Thus a sequence of trades is taken to measure the independent temporal parameter of market T, and is mapped onto the variable Θ, defined by integers ρ: $\Theta_\rho = \rho\tau + \Theta_0$.

Another reasonable scaling of t onto a mesoscopic Θ' would be to scale t inversely to volume V being traded, and to perform trades over a uniform mesh of Θ'. This would be one way of simulating an ‘‘average’’ trader.

(D) Price correlation studies imply that day trading as a speculative strategy is doomed to failure [3,21]. Some trends in markets no doubt exist, if only as partially self-fulfilling prophecies of regularly published forecast trends, albeit their net effect may also be to randomize the system away from these deterministic trends [22]. However, new unpredictable events and reactions to them, of course characteristic of the future, cause fluctuations to the extent that, in the absence of privileged information, losses and commissions usually take their final toll. The compelling conclusion is that, even with some trends present in markets, reactions to random events typically wipe out any correlations that might be used for successful day trading speculation. Therefore, perhaps if a sufficient set of variables directly related to variables actually used by traders are chosen to model these markets, then an underlying Markovian process, albeit highly nonlinear and nonequilibrium, would become apparent.

(E) Several pricing models have examined effects of nonlinear means and nonconstant variances, and have also attempted to understand how macroscopic markets interact with their microscopic agents [23,24]. However, after formulating their systems via continuous time generalizations to Eq. (1), errors are made in subsequent development of these equations to Fokker-Planck or Lagrangian formulations, especially in developing variational principles. An important issue concerns how or when to transform between the Ito and the Stratonovich representations of stochastic equations [25]. One main purpose of this paper is to point out the correct development, aided with the hindsight of similar developments in statistical mechanics within the past several years. A true Lagrangian is derived that replaces the ‘‘ ‘derived’ utility of wealth function’’ [24].

(F) Quite recent developments in nonlinear nonequilibrium statistical mechanics and their application to a variety of testable physical phenomena illustrate the importance of properly treating nonlinearities and nonequilibrium in systems where simpler analyses prototypical of linear equilibrium Brownian motion do not suffice [17-20]. It seems appropriate to at least give a fair test of these modern methods to the study of markets to address the previous features (A), (B), (C), (D), and (E).

Section 2 develops the formalism satisfying the requirements of feature (F). Section 3 develops a specific microscopic model of individual agents operating on a market. This example serves to explicitly demonstrate how the path integral formalism developed in Section 2 and in the Appendix can be identified with its underlying microscopic dynamics, especially in the ‘‘thermodynamic’’ limit. This formalism has not been previously used in the financial and economics literature, and only recently has it been applied to specific physical systems [18-20,26,27].

2. STATISTICAL DEVELOPMENT

When other order parameters in addition to price are included to study markets, Eq. (1) is accordingly generalized to a set of Stratonovich Langevin equations.

$$\dot{M}^G = f^G + \hat{g}^G_j \eta^j \ , \tag{2}$$

$$(G = 1, \cdots, \Lambda) \ ,$$

$$(j = 1, \cdots, N) \ ,$$

$$\dot{M}^G = \mathrm{d}M^G/\mathrm{d}\Theta \ ,$$

$$< \eta^j(\Theta) >_\eta = 0 \ ,$$

$$< \eta^j(\Theta), \eta^{j'}(\Theta') >_\eta = \delta^{jj'} \delta(\Theta - \Theta') \ ,$$

where f^G and $\hat{g}^G_j$ are generally nonlinear functions of mesoscopic order parameters M^G, j is a microscopic index indicating the source of fluctuations, and $N \geq \Lambda$. [See the Appendix for further specification of Eq. (2).] The Einstein convention of summing over repeated indices is used. Vertical bars on an index, e.g., |j|, imply no sum is to be taken on repeated indices. For example, consider mesoscopic market variable M^1 to be the published price at any time, and $\hat{g}^1_{|j|}\eta^{|j|}$ to be the stochastic driving influence on M^1 from a given trader j; this is derived from Eq. (1) by the simple change of variables $\Gamma = \log(M^1/\bar{M}^1)$. Alternately, one could consider that M^1 represents the attempted behavior/goal of agent j to follow the market price in the presence of all agents $j' = 1, \cdots, N$.

Via a somewhat lengthy, albeit instructive calculation, outlined in the Appendix, involving an intermediate derivation of a corresponding Fokker-Planck or Schrödinger-type equation for the conditional probability distribution $P[M(\Theta)|M(\Theta_0)]$, the Langevin rate Eq. (2) is developed into the more useful probability distribution for M^G at long-time macroscopic time event $\Theta = (u+1)\theta + \Theta_0$, in terms of a Stratonovich path-integral over mesoscopic Gaussian conditional probabilities [28-32]. Here, macroscopic variables are defined as the long-time limit of the evolving mesoscopic system.

The corresponding Schrödinger-type equation is [30,31]

$$\partial P/\partial\Theta = \frac{1}{2}(g^{GG'}P)_{,GG'} - (g^G P)_{,G} + V \ , \qquad (3)$$

$$g^{GG'} = k_T \delta^{jk} \hat{g}^G_j \hat{g}^{G'}_k \ ,$$

$$g^G = f^G + \frac{1}{2}\delta^{jk}\hat{g}^{G'}_j \hat{g}^G_{k,G'} \ ,$$

$$[\cdots]_{,G} = \partial[\cdots]/\partial M^G \ .$$

This is properly referred to as a Fokker-Planck equation when $V \equiv 0$. Note that although the partial differential Eq. (3) contains equivalent information regarding M^G as in the stochastic differential Eq. (2), all references to j have been properly averaged over. I.e., $\hat{g}^G_j$ in Eq. (2) is an entity with parameters in both microscopic and mesoscopic spaces, but M is a purely mesoscopic variable, and this is more clearly reflected in Eq. (3).

In the context of option pricing, several approaches [11,24,33,34] have derived a univariate Schrödinger-type equation with form similar to Eq. (3): Formally take M^1 = price, P = the rational call price, $g^{11} = (\sigma M^1)^2$, $g^1 = (2\sigma^2 - r)M^1$, $V = (\sigma^2 - 2r)$, where σ and r are empirical constants related to the variance of $\dot{M}^1/M^1$ and the short-term interest rate (equivalent to the risk in an efficient market).

The path integral representation is given in terms of the Lagrangian L.

$$P[M_\Theta|M_{\Theta_0}]dM(\Theta) = \int \cdots \int \underline{D}M \exp(-S)\delta[M(\Theta_0) = M_0]\delta[M(\Theta) = M_\Theta] \ , \qquad (4)$$

$$S = k_T^{-1} \min \int_{\Theta_0}^{\Theta} d\Theta' L \ ,$$

$$\underline{D}M = \lim_{u\to\infty} \prod_{\rho=1}^{u+1} g^{1/2} \prod_G (2\pi\theta)^{-1/2} dM^G_\rho \ ,$$

$$L(\dot{M}^G, M^G, \Theta) = \frac{1}{2}(\dot{M}^G - h^G)g_{GG'}(\dot{M}^{G'} - h^{G'}) + \frac{1}{2}h^G_{\ ;G} + R/6 - V \ ,$$

$$h^G = g^G - \frac{1}{2}g^{-1/2}(g^{1/2}g^{GG'})_{,G'} \ ,$$

$$g_{GG'} = (g^{GG'})^{-1} \ ,$$

$$g = \det(g_{GG'}) \ ,$$

$$h^G{}_{;G} = h^G_{,G} + \Gamma^F_{GF} h^G = g^{-1/2}(g^{1/2} h^G)_{,G} \ ,$$

$$\Gamma^F_{JK} \equiv g^{LF}[JK, L] = g^{LF}(g_{JL,K} + g_{KL,J} - g_{JK,L}) \ ,$$

$$R = g^{JL} R_{JL} = g^{JL} g^{JK} R_{FJKL} \ ,$$

$$R_{FJKL} = \frac{1}{2}(g_{FK,JL} - g_{JK,FL} - g_{FL,JK} + g_{JL,FK}) + g_{MN}(\Gamma^M_{FK}\Gamma^N_{JL} - \Gamma^M_{FL}\Gamma^N_{JK}) \ .$$

Mesoscopic variables have been defined as M^G in the Langevin and Fokker-Planck representations, in terms of their development from the microscopic system labeled by j. The Riemannian curvature term R arises from nonlinear $g_{GG'}$, which is a bona fide metric of this parameter space [30]. Even if a stationary solution, i.e., $\dot{M}^G = 0$, is ultimately sought, a necessarily prior stochastic treatment of $\dot{M}^G$ terms gives rise to these Riemannian "corrections." Even for a constant metric, the term $h^G{}_{;G}$ contributes to L for a nonlinear mean h^G. V may include terms such as $\sum_{T'} J_{T'G} M^G$, where the Lagrange multipliers $J_{T'G}$ are constraints on M^G, e.g., from other markets T', which are advantageously modelled as extrinsic sources in this representation; they too may be time-dependent. Using the variational principle below, J_{TG} may also be used to constrain M^G to regions where they are empirically bound. More complicated constraints may be affixed to L using methods of optimal control theory [35].

With respect to a steady state $\bar{P}$, when it exists, the information gain in state P is defined by

$$\Upsilon[P] = \int \cdots \int \underline{D}M' \, P \ln(P/\bar{P}) \ , \tag{5}$$

$$\underline{D}M' = \underline{D}M / dM_{u+1} \ .$$

In the economics literature, there appears to be sentiment to define Eq. (2) by the Ito, rather than the Stratonovich prescription. It should be noted that virtually all investigations of other physical systems, which are also continuous time models of discrete processes, conclude that the Stratonovich interpretation coincides with reality, when multiplicative noise with zero correlation time, modelled in terms of white noise η^j, is properly considered as the limit of real noise with finite correlation time [36]. The path integral succinctly demonstrates the difference between the two: The Ito prescription corresponds to the prepoint discretization of L, wherein $\theta\dot{M}(\Theta) \to M_{\rho+1} - M_\rho$ and $M(\Theta) \to M_\rho$. The Stratonovich prescription corresponds to the midpoint discretization of L, wherein $\theta\dot{M}(\Theta) \to M_{\rho+1} - M_\rho$ and $M(\Theta) \to \frac{1}{2}(M_{\rho+1} + M_\rho)$. In terms of the functions appearing in the Fokker-Planck Eq. (3), the Ito prescription of the prepoint discretized Lagrangian, L_I, is relatively simple, albeit deceptively so because of its nonstandard calculus.

$$L_I(\dot{M}^G, M^G, \Theta) = \frac{1}{2}(\dot{M}^G - g^G) g_{GG'} (\dot{M}^{G'} - g^{G'}) - V \ . \tag{6}$$

In the absence of a nonphenomenological microscopic theory, if the Ito prescription is proposed rather than the Stratonovich prescription, then this choice must be justified by numerical fits to data for each case considered. Differences between L and L_I have been found to be important in at least two physical systems investigated with these methods [18-20,26,27,37,38].

There are several other advantages to Eq. (4) over Eq. (2). Extrema and most probable states of M^G, $\ll M^G \gg$, are simply derived by a variational principle, similar to conditions sought in previous studies [24]. In the Stratonovich prescription, necessary, albeit not sufficient, conditions are given by

$$\delta_G L = L_{,G} - L_{,\dot{G}:\Theta} = 0 \ , \tag{7}$$

$$L_{,\dot{G}:\Theta} = L_{,\dot{G}G'} \dot{M}^{G'} + L_{,\dot{G}\dot{G}'} \ddot{M}^{G'} \ .$$

For stationary states, $\dot{M}^G = 0$, and $\partial\bar{L}/\partial\bar{M}^G = 0$ defines $\ll \bar{M}^G \gg$, where the bars identify stationary variables; in this case, the macroscopic variables are equal to their mesoscopic counterparts. [Note that $\bar{L}$ is *not* the stationary solution of the system, e.g., to Eq. (3) with $\partial P/\partial\Theta = 0$. However, in some cases [37], $\bar{L}$ is a definite aid to finding such stationary states.] Typically, in other financial studies, only properties of stationary states are examined, but here a temporal dependence is included. E.g., the $\dot{M}^G$ terms in L permit steady states and their fluctuations to be investigated in a nonequilibrium context. Note that Eq. (7) must be derived from the path integral, Eq. (4), which is at least one reason to justify its development.

In the language of nonlinear nonequilibrium thermodynamics [39], the thermodynamic forces are $\chi_G = S_{,G}$, where S is the entropy, and the thermodynamic fluxes are $\dot{M}^G = g^{GG'} \chi_{G'}$. Although the fluxes are defined to be linearly related to the forces, $g^{GG'}$ may be highly nonlinear in the state-variables M^G. The short-time Feynman Lagrangian L can be expressed as the sum of the dissipation function $\phi(M^G, \dot{M}^G) = \frac{1}{2} g_{GG'} \dot{M}^G \dot{M}^{G'}$, the force function $\Psi(M^G, \chi_G) = \frac{1}{2} g^{GG'} \chi_G \chi_{G'}$, the potential term $-V$, and the (negative) rate of change of entropy $-\dot{S}(M^G) = -\chi_G \dot{M}^G$: Then $L = \phi + \Psi - \dot{S} - V$ is the nonlinear nonequilibrium generalization of the Onsager-Machlup Lagrangian [40]. The variational equations insure that the equilibrium entropy is maximal, not necessarily a static equilibrium. Fluctuations over short time periods are introduced via variables $\eta_G = \partial L/\partial\dot{M}^G$ canonical to M^G, $\eta_G = g_{GG'}(\dot{M}^{G'} - g^{G'}) \equiv g_{GG'} \dot{M}^{G'} - \chi_G$, interpreted as resulting from the nonequilibrium competition between the thermodynamic forces and fluxes. In the context of multiplicative Gaussian noise, the conditional probability of making the state-transition from $M^G_\Theta = M^G(\Theta)$ to $M^G_{\Theta+\theta} = M^G(\Theta + \theta)$ is then hypothesized to be $P[M^G_{\Theta+\theta} | M^G_\Theta] \propto \exp(-\frac{1}{2k_T} \int_\Theta^{\Theta+\theta} d\Theta' L) d\eta$. This machinery suffices to determine the macroscopic probability distribution [39]. For nonconstant $g_{GG'}$ when $R \neq 0$, it should be noted that the Lagrangian corresponding to the most-probable path is not derived from the variational principle, but is directly related to L [41].

To begin introducing economic theory, variables such as (logarithm) price can be postulated to be the basic state-variables. However, it is not clear how to precisely relate L to classical economic equilibrium utility functions. It seems more reasonable to take economic microscopic models, usually formulated by differential equations of the state-variables, find regions of M^G wherein multiplicative Gaussian noise modelling is appropriate, directly calculate L as outlined in the Appendix, and then to make the identification with thermodynamic forces, fluxes and entropy, if this is desired. It is argued here that, although the thermodynamic interpretation perhaps has aesthetic value, the prime utility of the statistical mechanical formulation of probability densities in terms of generalized Lagrangians is that detailed calculations can be performed of macroscopic evolutions of microscopic and mesoscopic mechanisms, even in highly nonlinear and nonequilibrium contexts. The next Section 3 presents a microscopic model formulated such that the Lagrangian can be calculated directly from microscopic transition probabilities.

In spite of the difficulties just previously discussed, in relating L to classical economic equilibrium functions, it is appealing to consider that L represents relative gains in assets due to trading on the empirical data, similar to arguments invoked to establish the ‘‘derived’’ utility of wealth function [24]. E.g., the minima of L could be considered to correspond to maxima of demand functions (in terms of observable goods, prices and wealth) derived from utility functions (preference orderings) of individuals: Over short intervals of time, efficient adjustments between buyers and sellers should give rise to maximally rational actions of traders which should be reflected in most probable market variables. Then, the fit to empirical data could be achieved by fitting coefficients of polynomial expansions about $\ll M^G \gg$.

$$h^G = X^G + X^G_{G'} \underline{M}^{G'} + X^G_{G'G''} \underline{M}^{G'} \underline{M}^{G''} + \cdots \,, \tag{8}$$

$$g_{GG'} = Y_{GG'} + Y_{GG'G''} \underline{M}^{G''} + Y_{GG'G''G'''} \underline{M}^{G''} \underline{M}^{G'''} + \cdots \,,$$

$$\underline{M}^G = M^G - \ll M^G \gg \,.$$

All contributions to L in Eq. (4) are expressed in terms of h^G, $g_{GG'}$ and $\underline{M}^G$, and their derivatives. Determination of extrema that are minima or maxima are most conveniently ascertained numerically during the fitting process. Higher order terms in the series in Eq. (8) must be examined to determine their regions of convergence relative to the region of convergence of L as a series in $\underline{M}^G$. However, since the Lagrangian now appears as a Pade rational function, its region of convergence is expected to be at least as large as the regions of convergence of the means and variances.

Admittedly, the argument relating L to changes in assets is weak, here as well as in other studies without a rigorous development of a variational principle. I.e., the scale of derivation of utility functions and the scale of equilibrium description of means and variances of market variables are, respectively, comparable to thermodynamically comparing average molecular kinetic energies of a microscopic ensemble and the macroscopic temperature (times Boltzmann's constant) of a large sample. In nonequilibrium nonlinear dynamical systems, a mesoscopic level of description is necessary to accurately describe more realistic complex behavior, the subject of nonequilibrium nonlinear statistical mechanics.

A direct method of fitting parameters in L is to extract cumulative moments of the empirical data and to fit these to the (first several) cumulative moments of M^G, using Monte Carlo calculations [19] of the path integral. A simpler, cruder fit also could be done by fitting the most probable path to the stochastic data, i.e., minimizing the Lagrangian, evaluated at the empirical data, as a function of the expansion coefficients of Eq. (8). In addition to fitting the multivariate means, this method could still determine the multivariate covariances up to a constant factor. At least, this approach more accurately describes the empirical data, thereby establishing realistic functions for future theoretic models to derive. If $V \equiv 0$, then the corresponding Langevin Eq. (2) can be used to fit the parameters of Eq. (8), calculating ensemble averages of sets of stochastic trajectories for M^G.

As real systems are typically nonlinear, this procedure most likely yields sets of extrema $\{ \ll M^G \gg \}$, i.e., L is at least quadratic in some of its variables. This must be viewed as a practical optimistic first step in mapping out the more general functional behavior of $L(\dot{M}^G, M^G, \Theta)$. Changing constraints $J_{T'G}$ in V can drive a market T to different local minima, and competition and fluctuation between minima having varying degrees of local stability can give rise to phenomena typically found in other nonlinear nonequilibrium systems, e.g. bifurcation, hysteresis, etc. [17-20,37].

This process, of essentially fitting empirical data to the specific functional form of L, insures the conceptual Markov interpretation most popularly understood via Eq. (2). Another interesting aspect of Eq. (4) applied to sets of markets, is that some sets may exhibit similar functional dependencies in h^G and $g_{GG'}$. Then, their relative scalings, $\{k_T\}$, give a measure of their relative volatilities.

3. MICROSCOPIC MODEL

Because P represents a *bona fide* conditional probability distribution, the path integral representation suggests an approach to a microscopic theory of market behavior. This also permits acquisition of the functions f^G and $\hat{g}^G_j$ in Eq. (2), or of g^G and $g_{GG'}$ in Eqs. (3) and (4).

Consider the conditional probability distribution, p_j, of an agent j operating on a given market. For simplicity, assume that at time $t + \tau$, j must decide whether to buy or sell a standard increment of the market, based only on the information of the total number of buyers, M^B, and sellers, M^S, at time t. For example, take

$$p_{\sigma_j} = \frac{\exp(-\sigma_j F_j)}{[\exp(F_j) + \exp(-F_j)]} \tag{9}$$

$$\approx \frac{1}{2}\,[1 - \mathrm{erf}(\sigma_j F_j \sqrt{\pi}/2)]\ ,$$

$$\sigma_j = \begin{cases} +1 & \text{buy } (j \in B) \text{ or sell } (j \in S) \\ -1 & \text{do not act ,} \end{cases}$$

$$p_+ + p_- = 1\ ,$$

$$F_j = F_j(M^G)\ ,$$

$$G = \{B, S\}\ .$$

F_j may be any reasonably well-behaved function of M^B and M^S, different for buyers, $F_{j \in B} \equiv F^B$, or sellers, $F_{j \in S} \equiv F^S$. For simplicity, no other j dependence is considered in this model, and F_j is considered to represent a ‘‘decision factor’’ representing a ‘‘typical’’ rational agent in the market. Note that the probability distribution selected for p_{σ_j} closely approximates the cumulative distribution of a normal random variable, i.e., the ‘‘erf’’ function [18]. A mathematically (only) similar model has been developed for a different physical problem, which also demonstrates how to develop a field theory if M^G is homogeneously distributed in other variables [18-20].

A simple example of F_j for agents following market trends is obtained from

$$F^G_{\mathrm{ex1}} = a^G M^- / N\ , \tag{10}$$

$$M^- = M^B - M^S\ ,$$

where a^G are constants, $a^B < 0$ and $a^S > 0$, for agents following the trends of the market. I.e., agent j acts according to a sigmoid distribution with respect to market trends: p_{σ_j} is concave with respect to gains, and convex with respect to losses [42], but note that this simplified assumption of decision under risk is undergoing revision [43]. For convenience, assume that the total numbers of *potential* buyers and sellers are each constants,

$$j_S = 1, \cdots, N^S\ , \tag{11}$$

$$j_B = 1, \cdots, N^B\ ,$$

$$N = N^B + N^S\ .$$

At any given time, any agent may belong to either pool of S or B. Alternatively, permitting long and short trading, each agent could always be both a potential seller and a potential buyer, albeit with different decision factors F^G consistent with a desired net expected gain; then $N^S = N^B = N/2$. If each agent is considering one unit of a market’s assets, then the following development becomes a microscopic model of the dynamics of the market’s volume.

Note that ideal equilibrium, presumably fixed by arbitrage in an efficient market, is determined by N^S/N^B such that the total value of assets sold equals the total value of assets bought. I.e., this simple illustrative model is one in which price arbitrage interactions are emulated by quantity or supply interactions. However, in nonequilibrium, the appropriate order parameters are M^S and M^B, and a multistable or metastable market may prevail. This model may also be construed as the construction of microscopic probability distributions for agents, whose means represent individual utility functions, modelled in terms of microscopic variables σ, which subsequently are statistically aggregated into a mesoscopic Lagrangian L expressed in terms of mesoscopic variables, simple sums of σ_j. This is accomplished in this simple model without having to resort to the thermodynamic (static and equilibrium) limit, as discussed previously after Eq. (7).

The ''joint'' probability distribution P, joint with respect to pools of all S and B agents, but conditional with respect to time evolution, is

$$P[M(t+\tau)|M(t)] = \prod_G^{B,S} P^G[M^G(t+\tau)|M^{\bar{G}}(t)] \tag{12}$$

$$= \sum_{\sigma_j=\pm 1} \delta(\sum_{j\in S} \sigma_j - N^S)\delta(\sum_{j\in B} \sigma_j - N^B) \prod_j^N p_{\sigma_j}$$

$$= \prod_G (2\pi)^{-1} \int dQ^G \exp[iM^G(t+\tau)Q^G]$$

$$\times \prod_{j\in G}^{N^G} \cosh\{F_j[M(t)] + iQ^G\}\mathrm{sech}\{F_j[M(t)]\}$$

$$= \prod_G (1+E^G)^{-N^G} \binom{N^G}{\lambda^G} (E^G)^{\lambda^G} ,$$

$$E^G = \exp(-2F^G) ,$$

$$\lambda^G = [[\frac{1}{2}(M^G(t+\tau) + N^G)]] ,$$

$$M = \{M^G\} ,$$

where $M^{\bar{G}}(t)$ represents contributions from both $G = S$ and $G = B$ at time t, and λ^G is defined as the greatest integer in the double brackets. For convenience only, $\sigma_j F_j$ was defined so that $M^G = 0$ is arbitrarily selected as a midpoint between agents acting and not acting on the market: $M^G = -N^G$ signifies all agents not acting, $M^G = N^G$ signifies all agents acting.

The mean and variance of this binomial distribution yields

$$< M^G(t+\tau) >= -N^G \tanh F^G , \tag{13}$$

$$< M^G(t+\tau)M^{G'}(t+\tau) > - < M^G(t+\tau) >< M^{G'}(t+\tau) >= \frac{1}{4}\delta^{GG'} N^G \mathrm{sech}^2 F^G .$$

For large N^G and large $N^G F^G$, this binomial distribution is asymptotically Gaussian. With equal liklihood throughout time τ, any of the N uncorrelated agents $\sigma_j(t)$ can contribute to change the mesoscopic means and fluctuations of uncorrelated agents $\sigma_j(t+\tau)$. Therefore, for $\theta \le \tau$, at least to resolution $\theta \ge \tau/N$ and to order θ/τ, it is reasonable to assume a change in means of $\theta \dot{M}^G = M^G(t+\theta) - M^G(t) \approx \theta g^G$ with variance θg^{GG}. Defining $P^\theta = P[M^G(t+\theta)|M^G(t)]$ as a Gaussian distribution similar to $P^\tau = P[M^G(t+\tau)|M^G(t)]$, P^θ satisfies the Markovian Chapman-Kolmogorov equation $P^{\theta+\theta'} = \int P^\theta P^{\theta'}$, consistent with considering P^τ to be Markovian and as evolving from P^θ. It is conjectured here that requiring M^G to be continuous, albeit not necessarily differentiable, and pre-point discretized P^τ to be Markovian, suffice to reasonably define P^θ at the mesoscopic scale for $\theta \le \tau$. E.g., the same result should be obtained if mesoscopic distributions P^θ of variables M^G were extracted after considering microscopic σ_j contributions to p to have a temporal distribution within τ, e.g., Poisson. Folding P^θ for many time increments of θ into a macroscopic long-time distribution, the prepoint discretized L_I of Eq. (6) is naturally and directly obtained, with

$$g^G = -\tau^{-1}(M^G + N^G \tanh F^G) \ , \tag{14}$$

$$g^{GG'} = \delta_G^{G'} \tau^{-1} N^G \mathrm{sech}^2 F^G \ .$$

It should be noted that any number of classes of agents may be specified in Eq. (9), e.g., to emulate coalitions of agents, as long as the number in each class is sufficient to permit the central limit theorem to be applied to obtain the Gaussian statistics defining L. For example, G might represent two classes of agents with different decision functions F^G, and σ_j might represent selling ($\sigma_j = +1$) or buying ($\sigma_j = -1$). Other or additional options could be specified for G, F^G and σ_j. Even for $|F^G(z)| \ll 1$, for large N^G, $N^G F^G$ will be large enough in some neighborhood of z to permit this statistical development.

Examining this derivation of $g^{GG'}$, and the derivation of the path integral from the corresponding Langevin equations, it is seen that in Eq. (2) a reasonable choice for $\hat{g}_j^G$ is

$$\hat{g}_j^G = \begin{cases} \delta_{G'}^G g^{GG'}/(N^G \tau) \ , & j \in G \\ 0 \ , & j \in G' \neq G \ , \end{cases} \tag{15}$$

This mesoscopic model is derived from a sigmoid distribution for an individual agent. However, some new structure typically appears as a result of the collective pool of agents: E.g., for example 1, it is clear that a stationary state, $\dot{M}^G = 0$, is near $\bar{M}^G = 0$. However, for F^G more nonlinear, there may be additional roots to $N^G = -\tanh F^G$, the Riemannian ‘‘corrections’’ may add significant structure, and competition between B and S may produce additional structure. All these effects are most succinctly understood by examining the Lagrangian. This directly illustrates the intuitive and analytic utility of the path integral. The existence and location of local extrema are not as easily found from the corresponding Langevin and Fokker-Planck equations.

Figure 1 presents 3-dimensional and contour plots over $\bar{M}^G$-space for $\bar{L}_{\mathrm{ex1}}$, where $\bar{L}_{\mathrm{ex1}}$ represents the stationary Lagrangian of Eq. (4) for the microscopic model in Eq. (10). Arbitrarily, take $N^G = 100$, $J_G = 0$, $a^B = -2$ and $a^S = 1$.

For comparison, and to see the size of the approximation made in using the Ito prescription, Fig. 2 gives the corresponding plots as in Fig. 1, but analyses $\bar{L}_{I\mathrm{ex1}}$ instead of $\bar{L}_{\mathrm{ex1}}$. There is barely a measurable difference, e.g., within 1%, for this simple example, but the multiple stationary minima are lost at this scale for the Ito prescription. In general, it may be expected that the more realistically nonlinear is F^G, the larger this difference will be.

As another example, consider in Eq. (9),

$$F_{\mathrm{ex2}}^G = \frac{a^G M^- / N + b^G}{1 + c^G [(M^B)^2 + (M^S)^2]/N^2} \ , \tag{16}$$

which represents a modification to the sigmoid distribution of example 1. Arbitrarily, $b^G = 0$ and $c^G = 1$. In Fig. 3, multiple stationary minima are still seen in the region of $\bar{M}^G$-space of concern to this model.

Figure 4 gives plots similar to Fig. 3, but using $\bar{L}_{I\mathrm{ex2}}$ instead of $\bar{L}_{\mathrm{ex2}}$ to calculate the behavior induced by the microscopic system of Eq. (16). The structure in Fig. 3 is clearly changed.

More complex decision factors F^G must be expected to produce even more structure in the stationary and dynamic behaviors of the market. Note that $J_{T'G}$ constraints, e.g., influences from other markets, advantageously model restrictions to neighborhoods of particular extrema, thereby stressing possibilities of interaction between nearby multiple minima.

One approach to bridge microscopic and macroscopic theory is to select a microscopic function F_j, enabling extrema $\ll M^G \gg$ to be solved for as solutions of the variational Eq. (7), developed from the path integral representation. Then the response of an agent in a neighborhood of $\ll M^G \gg$ can be further explored using p_{σ_j} in Eq. (9), even to determine a more suitable starting function F_j.

Alternatively, in an opposite approach, mesoscopic extrema $\ll M^G \gg$ can be found from empirical fits to the data, as described by Eq. (8). In the neighborhood of $\ll M^G \gg$, the average agent's contribution to the Langevin representation can be calculated by Eq. (15). Furthermore, Eq. (14) may be solved to find F^G, thereby determining F_j in this neighborhood. This method acts to determine the microscopic distribution from the macroscopic one.

4. DISCUSSION

A nonlinear nonequilibrium statistical mechanics description of markets is presented in a form immediately operational and calculable. Given are the primary equations necessary to establish the theoretical context and to perform explicit fits on empirical data. This approach puts the statistical analysis of markets into a current paradigm being applied to other nonlinear nonequilibrium systems.

The Lagrangian derived is a concise fundamental description of empirical data, superior to mere curve fitting to lay a foundation upon which to further investigate economic mechanisms responsible for dynamical functional relationships among market variables. The Lagrangian is *a priori* constrained to consistently include effects of nonlinearity and nonequilibrium in probability distributions of several or many variables.

A simple model is presented, by which the derivation of a *bona fide* probability distribution further enables the development of specific microscopic economic models, themselves described by probability distributions of operations of individual agents. Although this microscopic model directly leads to the path integral, it has also been demonstrated how these results are directly translated into Fokker-Planck and Langevin languages typically used by other investigators. This model shows how a bridge might be made from microscopic to macroscopic theory, especially in the thermodynamic (static and equilibrium) limit. However, the formalism presented here is more directly and generally applicable to a realizable mesoscopic dynamical statistical analysis of macroscopic markets, given the realistic constraints of the probable existence of rather complex microscopic behavior.

The variational principle possessed by the Lagrangian is an important tool to further study this interaction. An explicit algorithm is thereby developed to study the interactions between macroscopic markets and individual agents, but at a mesoscopic scale between the two.

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APPENDIX

This Appendix outlines the derivation of the path integral representation of the nonlinear Langevin equations, via the Fokker-Planck representation. This serves to point out the importance of properly treating nonlinearities, and to emphasize the deceptive simplicity of the Langevin and Fokker-Planck representations of stochastic systems. There are a few derivations in the literature, but the following blend seems to be the most concise. All details may be found in the references given in this paper [28,31,32,41].

The Stratonovich Langevin equations given in Eq. (2) can be analyzed in terms of the Wiener process $\mathrm{d}W^i$, which can be rewritten in terms of Gaussian noise $\eta^i = \mathrm{d}W^i/\mathrm{d}t$ if care is taken in the limit [41].

$$\mathrm{d}M^G = f^G[t, M(t)]\mathrm{d}t + \hat{g}_i^G[t, M(t)]\mathrm{d}W^i \ , \tag{A1}$$

$$\dot{M}^G(t) = f^G[t, M(t)] + \hat{g}_i^G[t, M(t)]\eta^i(t) \ ,$$

$$dW^i \to \eta^i dt \ ,$$

$$M = \{M^G; G = 1, \cdots, \Lambda\} \ ,$$

$$\eta = \{\eta^i; i = 1, \cdots, N\} \ .$$

η^i represents Gaussian white noise, and moments of an arbitrary function $F(\eta)$ over this stochastic space are defined by a path-type integral over η^i,

$$< F(\eta) >_\eta = \bar{N}^{-1} \int D\eta F(\eta) \exp(-\frac{1}{2} \int_{t_0}^{\infty} dt \eta^i \eta^i \ ) \ , \tag{A2}$$

$$\bar{N} = \int D\eta \exp(-\frac{1}{2} \int_{t_0}^{\infty} dt \eta^i \eta^i \ ) \ ,$$

$$D\eta = \lim_{v \to \infty} \prod_{\alpha=0}^{v+1} \prod_{j=1}^{N} (2\pi\theta)^{-1/2} dW_\alpha^j \ ,$$

$$t_\alpha = t_0 + \alpha\theta \ ,$$

$$\frac{1}{2} \int dt \eta^i \eta^i = \frac{1}{2\theta} \sum_\beta \sum_i (W_\beta^i - W_{\beta-1}^i)^2 \ ,$$

$$< \eta^i >_\eta = 0 \ ,$$

$$< \eta^i(t)\eta^j(t') >_\eta = \delta^{ij}\delta(t - t') \ ,$$

where $t \to \Theta$ in the text.

Non-Markovian sources, $\hat{\eta}$, and their influence throughout this development, can be formally treated by expansions about the Markovian process by defining

$$< F(\hat{\eta}) >_\eta = \bar{N}_\xi^{-1} \int D\hat{\eta} F \ \exp[-\frac{1}{2} \int \int dt dt' \hat{\eta}(t) \Delta_\xi^{-1}(t - t') \hat{\eta}(t')] \ , \tag{A3}$$

$$\int dt \ \Delta_\xi^{-1}(t - t') \Delta_\xi(t' - t'') = \delta(t - t'') \ ,$$

with ξ defined as an interval centered about the argument of Δ_ξ. Letting $\xi \to 0$ is an unambiguous procedure to define the Stratonovich prescription used below.

In terms of a specific stochastic path η, a solution to Eq. (A1), $M_\eta^G(t; M_0, t_0)$ with $M_\eta^G(t_0; M_0, t_0) \equiv M_0$, the initial conditions on the probability distribution of M_η is

$$P_\eta[M, t|M_0, t_0] = \delta[M - M_\eta(t; M_0, t_0)] \ . \tag{A4}$$

Using the conservation of probability condition,

$$P_{\eta,t} + (\dot{M}^G P_\eta)_{,G} = 0 \ , \tag{A5}$$

$$[\cdots]_{,G} = \partial[\cdots]/\partial M^G \ ,$$

$$[\cdots]_{,t} = \partial[\cdots]/\partial t \ ,$$

the evolution of P_η is written as

$$P_{\eta,t}[M,t|M_0,t_0] = \{[-f^G(t,M) - \hat{g}(t,M)\eta^i]P_\eta\}_{,G} \ . \tag{A6}$$

To perform the stochastic average of Eq. (A6), the ‘‘functional integration by parts lemma’’ [44] is used on an arbitrary function $Z(\eta)$ [31],

$$\int D\eta \, \frac{\hat{\delta} Z(\eta)}{\hat{\delta}\eta^i} = 0 \ . \tag{A7}$$

Applied to $Z = Z' \exp(-\frac{1}{2}\int_{t_0}^{\infty} \mathrm{d}t \eta^i \eta^i)$, this yields

$$< \eta^i Z' >_\eta = < \delta Z'/\delta\eta^i >_\eta \ . \tag{A8}$$

Applying this to $\hat{F}[M_\eta] = \int \mathrm{d}M \ P_\eta F(M)$,

$$\int \mathrm{d}M \, \frac{\hat{\delta} P_\eta}{\hat{\delta}\eta^i} F(M) = \frac{\partial \hat{F}[M_\eta]}{\partial M_\eta^G} \frac{\hat{\delta} M_\eta^G}{\hat{\delta}\eta^i} \tag{A9}$$

$$= -\frac{1}{2}\int \mathrm{d}M \ F(M)(\hat{g}_j^G \delta^{ij} P^\eta)_{,G} \ ,$$

where $\hat{\delta}$ designates functional differentiation. The last equation has used the Stratonovich prescription,

$$M_\eta^G(t) = M_0^G + \int \mathrm{d}t' \hat{H}(t-t')\hat{H}(t-t_0)(f^G + \hat{g}_i^G \eta^i) \ , \tag{A10}$$

$$\lim_{t' \to t-0} \frac{\hat{\delta} M_\eta^G(t)}{\hat{\delta}\eta^i(t')} = \frac{1}{2}\hat{g}_j^G[t, M_\eta(t)]\delta_{ij} \ ,$$

$$\hat{H}(z) = \begin{cases} 1, \ z \geq 0 \\ 0, \ z < 0 \ . \end{cases}$$

Taking the averages $< P_{\eta,t} >_\eta$ and $< \eta^i P_\eta >_\eta$, the Fokker-Planck is obtained from Eq. (A9). If some boundary conditions are added as Lagrange multipliers, these enter as a ‘‘potential’’ V, creating the Schrödinger-type equation given in Eq. (3) in the text, with $\Theta \to t$ here.

$$P_{,t} = \frac{1}{2}(g^{GG'}P)_{,GG'} - (g^G P)_{,G} + VP \ , \tag{A11}$$

$$P = < P_\eta >_\eta \ ,$$

$$g^G = f^G + \frac{1}{2}\hat{g}_i^{G'}\hat{g}_{i,G'}^G \ ,$$

$$g^{GG'} = \hat{g}_i^G \hat{g}_i^{G'} \ .$$

Note that g^G replaces f^G in Eq. (A1) if the Ito calculus is used to define that equation.

To derive the path integral representation of Eq. (A11), define operators $\hat{M}^G$, $\hat{p}_G$ and $\hat{H}$,

$$[\hat{M}^G, \hat{p}_{G'}] \equiv \hat{M}^G \hat{p}_{G'} - \hat{p}_{G'}\hat{M}^G = i\delta_{G'}^G \ , \tag{A12}$$

$$[\hat{M}^G, \hat{M}^{G'}] = 0 = [\hat{p}_G, \hat{p}_{G'}] ,$$

$$P_{,t} = -i\hat{H}P ,$$

$$\hat{H} = -\frac{i}{2}\,\hat{p}_G\hat{p}_{G'}g^{GG'} + \hat{p}_G g^G + iV ,$$

and define the evolution operator $U(t, t')$ in terms of ''bra'' and ''ket'' probability states of M,

$$\hat{M}^G|M^G> = M^G|M^G> , \tag{A13}$$

$$\hat{p}_G|M^G> = -i\partial/\partial M^G|M^G> ,$$

$$< M'|M> = \delta(M' - M) ,$$

$$< M|p> = (2\pi)^{-1}\exp(ip \cdot M) ,$$

$$P[M, t|M_0, t_0] = < M|U(t, t_0)|M_0> ,$$

$$\hat{H}(t')U(t', t) = iU(t', t)_{,t'} ,$$

$$U(t, t) = 1 ,$$

$$U(t_\rho, t_{\rho-1}) \approx 1 - i\theta\hat{H}(t_{\rho-1}) ,$$

where ρ indexes units of θ measuring the time evolution. This is formally integrated to give the path integral in the phase space (p, M),

$$P[M_t|M_0] = \int\limits_{M(t_0)=M_0}^{M(t)=M_t} DM\ Dp\ \exp[\int\limits_{t_0}^{t} dt'(ip_G M^G - \frac{1}{2}\,p_G p_{G'}g^{GG'} - ip_G g^G + V)] ,$$

$$DM = \lim_{u\to\infty}\prod_G\prod_{\rho=1}^{u} dM_\rho^G , \tag{A14}$$

$$Dp = \lim_{u\to\infty}\prod_G\prod_{\rho=1}^{u+1}(2\pi)^{-1}dp_{G\rho} ,$$

$$t_\rho = t_0 + \rho\theta .$$

The integral over each $dp_{G\rho}$ is a Gaussian and simply calculated. This gives the path integral in coordinate space M, in terms of the prepoint discretized Lagrangian, also defined by Eq. (6),

$$P[M_t|M_0] = \int DM\prod_{\rho=0}^{u}(2\pi\theta)^{-\Lambda/2}g(M_\rho, t_\rho)^{1/2} \tag{A15}$$

$$\times\exp\{-\frac{1}{2}\,\theta g_{GG'}(M_\rho, t_\rho)[\Delta_\rho^G/\theta - g^G(M_\rho, t_\rho)]$$

$$\times[\Delta_\rho^{G'}/\theta - g^{G'}(M_\rho, t_\rho)] + \theta V(M_\rho, t_\rho)\} ,$$

$$g = \det(g_{GG'}) \ ,$$

$$g_{GG'} = (g^{GG'})^{-1} \ ,$$

$$\Delta^G_\rho = M^G_{\rho+1} - M^G_\rho \ .$$

This can be transformed to the Stratonovich representation, in terms of the Feynman Lagrangian L possessing a variational principle,

$$P[M_t|M_0] = \int DM \prod_{\rho=0}^{u} (2\pi\theta)^{-\Lambda/2} g(M_\rho + \Delta_\rho, t_\rho + \theta/2)^{1/2} \tag{A16}$$

$$\times \exp\{ -\min \int_{t_\rho}^{t_\rho+\theta} dt' L[M(t'), \dot{M}(t'), t'] \} \ ,$$

where "min" specifies that Eq. (A11) is obtained by constraining L to be expanded about that $M(t)$ which makes the action $S = \int dt' L$ stationary for $M(t_\rho) = M_\rho$ and $M(t_\rho + \theta) = M_{\rho+1}$. One way of proceeding is to expand Eq. (A15) and compare to Eq. (A16), but it is somewhat easier to expand Eq. (A16) and compare to Eq. (A15) [32]. It can be shown that expansions to order θ suffice, and that $\Delta^2 = O(\theta)$. (For convenience, the constant k_T appearing in Eq. (4) is set equal to unity until the end of the derivation. This would appear as a $k_T^{1/2}$ factor of $\hat{g}^G_i$ in Eqs. (A1) and (A11), also yielding a k_T factor of $g^{GG'}$ in Eq. (A11), thereby scaling the Lagrangian of the path integral by k_T^{-1}.)

Write L in the general form

$$L = \frac{1}{2} g_{GG'} \dot{M}^G \dot{M}^{G'} - h_G \dot{M}^G + b \tag{A17}$$

$$= L^0 + \Delta L \ ,$$

$$L^0 = \frac{1}{2} g_{GG'}[M(t), t] \dot{M}^G \dot{M}^{G'} \ ,$$

$$g_{GG'}[M(t), t] = g_{GG'}[M(t), t'] + g_{GG',t'}[M(t), t'](t - t') + O[(t - t')^2] \ ,$$

where h_G and b must be determined by comparing expansions of Eqs. (A15) and (A16). Only the L^0 term is dependent on the actual $M(t)$ trajectory, and so

$$\int_{t_\rho}^{t_\rho+\theta} dt \ \Delta L = (\frac{1}{4} g_{GG',t} \Delta^G \Delta^{G'} - h_G \Delta^G - \frac{1}{2} h_{G,G'} \Delta^G \Delta^{G'} + \theta b)|_{(M,t)} \ , \tag{A18}$$

where "$|_{(M,t)}$" implies evaluation at (M, t).

The determinant g is expanded as

$$g(M + \Delta, t + \theta/2)^{1/2} \approx g^{1/2}(M, t) \exp[\frac{\theta}{4g} g_{,t} + \frac{1}{2g} \Delta^G g_{,G} \tag{A19}$$

$$+ \frac{1}{4g} \Delta^G \Delta^{G'} (g_{,GG'} + g^{-1} g_{,G} g_{,G'})]|_{(M,t)} \ .$$

The remaining integral over L^0 must be performed. This is accomplished using the variational principle applied to $\int L^0$ [28],

$$g_{GH}\ddot{M}^H = -\frac{1}{2}(g_{GH,K} + g_{GK,H} - g_{KH,G})\dot{M}^K\dot{M}^H \ , \tag{A20}$$

$$\ddot{M}^F = -\Gamma^F_{JK}\dot{M}^J\dot{M}^K \ ,$$

$$\Gamma^F_{JK} == g^{LF}[JK,L] = g^{LF}(g_{JL,K} + g_{KL,J} - g_{JK,L}) \ ,$$

$$(\frac{1}{2}g_{GH}\dot{M}^G\dot{M}^H)_{,t} = 0 \ ,$$

$$\int_t^{t+\theta} L^0 dt \approx \frac{\theta}{2} g_{GH}\dot{M}^G\dot{M}^H|_{(M,t+\theta)} \ .$$

Differentiating the second equation in Eq. (A20) to obtain $\dddot{M}$, and expanding $\dot{M}(t+\theta)$ to third order in θ,

$$\dot{M}(t+\theta) = [\frac{1}{\theta}\Delta^G - \frac{1}{2\theta}\Gamma^G_{KL}\Delta^K\Delta^L + \frac{1}{6\theta}(\Gamma^G_{KL,N} + \Gamma^G_{AN}\Gamma^A_{KL})\Delta^G\Delta^L\Delta^N]|_{(M,t)} \ . \tag{A21}$$

Now Eq. (A16) can be expanded as

$$P[M_t|M_0]dM(t) = \int \underline{D}M \prod_{\rho=0}^{u} \exp[-\frac{1}{2\theta}g_{GG'}(M,t)\Delta^G\Delta^{G'} + B] \ , \tag{A22}$$

$$\underline{D}M = \prod_{\rho=1}^{u+1} g_\rho^{1/2} \prod_G (2\pi\theta)^{-1/2} dM^G_\rho \ .$$

Expanding $\exp B$ to $O(\theta)$ requires keeping terms of order Δ, Δ^2, Δ^3/θ, Δ^4/θ, and Δ^6/θ^2. Under the path integral, evaluated at (M,t), and using "$\doteq$" to designate the order of terms obtained from $\int d\Delta\,\Delta^n \exp(-\frac{1}{2\theta}\Delta^2)$,

$$\Delta^G\Delta^H \doteq \theta g^{GH} \ , \tag{A23}$$

$$\Delta^G\Delta^H\Delta^K \doteq \theta(\Delta^G g^{HK} + \Delta^H g^{GH} + \Delta^K g^{GH}) \ ,$$

$$\Delta^G\Delta^H\Delta^A\Delta^B \doteq \theta^2(g^{GH}g^{AB} + g^{GA}g^{HB} + g^{GB}g^{HA}) \ ,$$

$$\Delta^A\Delta^B\Delta^C\Delta^D\Delta^E\Delta^F \doteq \theta^3(g^{AB}g^{CD}g^{EF} + 14 \text{ permutations}) \ .$$

This expansion of $\exp B$ is to be compared to Eq. (A15), expanded as

$$P[M_t|M_0]dM(t) \approx \int \underline{D}M \prod_{\rho=0}^{u} \exp(-\frac{1}{2\theta}g_{GG'}\Delta^G\Delta^{G'}) \tag{A24}$$

$$\times[1 + g_{GG'}g^G\Delta^{G'} + \theta V + O(\theta^{3/2})] \ ,$$

yielding identification of h_G and b in Eq. (A17),

$$h^G = g^{GG'}h_{G'} = g^G - \frac{1}{2}g^{-1/2}(g^{1/2}g^{GG'})_{,G'} \ , \tag{A25}$$

$$b = \frac{1}{2}h^G h_G + \frac{1}{2}h^G{}_{;G} + R/6 - V \ ,$$

$$h^G{}_{;G} = h^G_{,G} + \Gamma^F_{GF} h^G = g^{-1/2} (g^{1/2} h^G)_{,G} \ ,$$

$$R = g^{JL} R_{JL} = g^{JL} g^{FK} R_{FJKL} \ .$$

This result gives Eq. (4) in the text, with $t \rightarrow \Theta$ and scaling L with k_T^{-1}.

$$P[M_t | M_0] dM(t) = \int \cdots \int \underset{\sim}{D} M \exp(-S) \ , \tag{A26}$$

$$S = k_T^{-1} \min \int_{t_0}^{t} dt' L \ ,$$

$$L(\dot{M}^G, M^G, t) = \frac{1}{2} (\dot{M}^G - h^G) g_{GG'} (\dot{M}^{G'} - h^{G'}) + \frac{1}{2} h^G{}_{;G} + R/6 - V \ .$$

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FIGURE CAPTIONS

Fig. 1. Example F^G_{ex1} of Section 3 is analyzed by examining the stationary Lagrangian $\tau\bar{L}_{\text{ex1}}$ as a function of its variables $|\bar{M}^G| \le N^G$. (a) is a 3-dimensional plot of the surface of $\tau\bar{L}_{\text{ex1}}$ over $\bar{M}^G$. The perspective is determined by smallest and largest values of $\tau\bar{L}$, $\tau\bar{L}_{\text{min}}$ and $\tau\bar{L}_{\text{max}}$, respectively, at points min = ($\bar{M}^B_{\text{min}}$, $\bar{M}^S_{\text{min}}$, $\tau\bar{L}_{\text{min}}$) and max = ($\bar{M}^B_{\text{max}}$, $\bar{M}^S_{\text{max}}$, $\tau\bar{L}_{\text{max}}$). The plots are projected onto a plane perpendicular to the line running between a point on the line of sight, chosen here to be (max + min)/2, and the point from which the projection is made, chosen here to be max +3(max − min). The horizontal $\bar{M}^S$ axis increases to the right, and the sloping $\bar{M}^B$ axis increases towards the left. (b) is a contour plot of (a). $\bar{M}^B$ on the horizontal axis increases to the right, and $\bar{M}^S$ is on the vertical axis increasing upwards. There exists an outermost completely closed contour at ~0.04. $\tau\bar{L}_{\text{min}} \sim -2.5 \times 10^{-3}$ near the zero contours.

Fig. 2. Plots (a) and (b) are similar to Fig. 1, except that $\tau\bar{L}_{I\text{ex1}}$ is analyzed instead of $\tau\bar{L}_{\text{ex1}}$. In (b), the outermost completely closed contour is also at ~0.04. $\tau\bar{L}_{\text{min}} \sim -2.5 \times 10^{-3}$ near the zero contours.

Fig. 3. Plots (a) and (b) correspond to those in Fig. 1, except that $\tau\bar{L}_{\text{ex2}}$ for example 2 is analyzed. In (b), the outermost completely closed contour is at ~0.06.

Fig. 4. Plots (a) and (b) correspond to those in Fig. 3, except that $\tau\bar{L}_{I\text{ex2}}$ is analyzed instead of $\tau\bar{L}_{\text{ex2}}$. In (b), the outermost completely closed contour is also at ~0.06.