**Physical Studies Institute LLC** $\psi$
**PhysicalStudiesInstitute.org**

L Ingber PhD ingber@caa.caltech.edu lester@ingber.com ingber.com

# Quantum Variables in Finance

**Lester Ingber**
**Physical Studies Institute LLC**
**Hillsboro, Oregon**

## ABSTRACT

**Background:** A path-integral algorithm, PATHINT used previously for several systems, has been generalized from 1 dimension to N dimensions, and from Classical to Quantum systems into qPATHINT. Previous publications applied qPATHINT to two systems developed by the author, in neocortical interactions and financial options. Classical PATHINT also has been published demonstrating development of Eurodollar options in industrial applications.

**Objective:** Historical Volatility (HstVol) or Implied Volatility (ImpVol) (preferred by traders) from Strike data are used and are required to price financial options. This proposal creates tables of ImpVol, first for Classical and Quantum old (circa 2000) Eurodollars which benchmarks all codes to previous publications, then for Quantum Bitcoin which is among the most volatile currencies. Only Classical (super)computers are required here for all calculations.

**Method:** The system is developed using mathematical-physics methods of path integrals in Quantum spaces. Supercomputer pilot studies using XSEDE.org and StonyBrook.edu Ookami resources tested various dimensions for their scaling limits. For this study, ImpVols and all traded Greeks are calculated for options in Quantum-money spaces, including realistic shocks (dividends, etc.).

**Results:** The mathematical-physics and computer parts of the study are successful for both systems. A 3-dimensional path-integral propagation of qPATHINT for both systems is within normal computational bounds on supercomputers, but here only 1-dimensional path-integrals are required.

**Conclusion:** Each of the two systems considered have contributed insight into applications of qPATHINT to the other system, leading to new algorithms presenting time-dependent propagation of interacting Quantum and Classical scales.

**Keywords:** Nonlinear Stochastic Systems, Quantum Systems, Importance-Sampling Optimization, Financial Markets

## 1. Executive Summary

Quantum computing is here, and in the near future this is expected to include financial products with blockchains. It not far-fetched to assume that soon derivatives will be developed on these products. As is the case in Classical real spaces with PATHTREE and PATHINT, qPATHTREE and qPATHINT are poised to calculate derivatives in Quantum complex spaces. This is beyond using Quantum computation of derivatives, since the space of the dependent variables themselves may live in Quantum worlds (Baaquie *et al*, 2002; Piotrowski *et al*, 2005; Accardi & Boukas, 2007; Meyer, 2009; Aaronson & Christiano, 2012; Jogenfors, 2016). Quantum and Classical are capitalized here if not in References.

This proposal puts equations in appendicies: Section 2 Appendix A is Previous Benchmarks. Section 3 Appendix B is Quantum Money. Section 4 Appendix C is Path Integrals. Section 5 Appendix D is Adaptive Simulated Annealing (ASA).

### 1.1. Path Integral Applications

Path integrals and PATHINT have been applied across several disciplines, e.g., in combat analyses (Ingber, Fujio & Wehner, 1991), neuroscience (Ingber, 1994; Ingber & Nunez, 1995; Ingber & Nunez, 2010; Ingber, 2017c), finance (Ingber & Wilson, 2000; Ingber, 2000; Ingber, Chen *et al*, 2001; Ingber, 2016; Ingber, 2017a; Ingber, 2017b; Ingber, 2017c), and other nonlinear systems (Ingber, 1995; Ingber, Srinivasan & Nunez, 1996; Ingber, 1998a).

### 1.2. Statistical Mechanics of Neocortical Interactions (SMNI)

Path integrals have been used by the author primarily in Statistical Mechanics of Neocortical Interactions (SMNI) studies. A StonyBrook.edu Ookami project, "Statistical Mechanics of Neocortical Interactions: Quantum influences on neocortical information processing," ongoing since 2012, has produced several relevant papers discussing the path-integral algorithm (Ingber, 2022b; Ingber, 2021b), including papers published at XSEDE.org under grants dating from February 2013 through December 2021 which contain more SMNI references (Ingber, 2018; Ingber, 2019; Ingber, 2021b; Ingber, 2021c; Ingber, 2022b). This work included the author's closed-form Quantum path-integal solutions for tripartite (neuron-astrocyte-neuron) interactions.

### 1.3. Statistical Mechanics of Financial Markets (SMFM)

Another project since 2017, using similar mathematical-physics tools, was ongoing at XSEDE.org until 2021, "Quantum Options," under Grant TG-MCB140110 (Ingber, 2017c; Ingber, 2021a).

The codes developed enable calculation of wave functions $\psi$ and moments of physical operators $O$, $< \psi^* |O| \psi >$. To numerically calculate the path integral, especially for serial changes in time — not approachable with standard Monte Carlo techniques — PATHINT was developed.

Note that the financial applications yet have no real-world applications, and likely will not for a decade when Quantum computers will become mainstream, and then Quantum money will become a reality, if no other reason than Quantum states cannot be cloned/counterfeited (Aaronson & Christiano, 2012; Jogenfors, 2016; Ingber, 2017b). At that time, to control volatility, options on Quantum money will be required, and the calculations performed by qPATHINT and qPATHTREE to calculate options present all Greeks necessary for trading.

Historical Volatility (HstVol) or Implied volatility (ImpVol) (preferred by traders) from Strike data are essential for trading options, i.e., setting fair prices for options (Ingber, Chen *et al*, 2001). This requires searching for minima in the nonlinear volatility variable fit to Strike data, which is most influential in options pricing. Since cryptocurrencies currently exhibit the largest swings in volatility (Zulfiqar & Gulzar, 2021) they will be fit after baselining the code to Eurodollars. Probabilities will be based on fits to seasonal real data, the expressions will be quite nonlinear in volatility, and so the author's ASA (under a public license and available to all without any registration at ingber.com) is used (Ingber, 1993a).

However, it is likely that not every fit requires the full power (and time) of ASA. As pointed in other publications (Ingber, 2020), often after a full ASA fit for the first of a given market, the Simplex code that comes with the ASA code (used efficiently to get the last few decimal places) can be used to modify the original fitted parameter(s).

The Classical counterparts to Quantum options have used path-integral methods successfully to calculate options with nonlinear and time-dependent multivariate options (Ingber, 2000).

### 1.4. Approximate State of Quantum Hybrid Computing

As of 2020 (Ingber, 2021c), there are several companies now offering commercial-grade Hybrid Classical-Quantum computers, e.g., Rigetti, D-Wave, Google, Microsoft, and IBM; see

https://docs.ocean.dwavesys.com/projects/hybrid/en/latest/index.html
https://www.rigetti.com/what
https://azure.microsoft.com/en-us/solutions/hybrid-cloud-app/#overview
https://www.ibm.com/it-infrastructure/z/capabilities/hybrid-cloud

Hybrid computers typically offer Classical computers to optimize parameters in systems that are described by Quantum variables using Quantum computers (Benedetti *et al*, 2019). Other studies show Quantum computing still out of reach for many systems, even with Classical optimizers (Chakrabarti *et al*, 2020). Software for Quantum states also has been developed, e.g., Tensorflow, the popular end-to-end open-source tool for machine learning, also has been adapted for Hybrid Classical-Quantum computing:

https://quantumzeitgeist.com/tensorflow-for-quantum-hits-first-birthday/
https://www.tensorflow.org/quantum

It has been noted that SMNI can offer more detail to many of these neural-net projects (Ingber, 2022c).

The PI has accounts on D-Wave and Rigetti computers, but has not yet ported SMFM code to these Quantum computers for Hybrid computations.

This project merges codes generated for two previous XSEDE grant proposals, "Electroencephalographic field influence on calcium momentum waves" (Ingber, 2018) and "Quantum path-integral qPATHTREE and qPATHINT algorithms" (Ingber, 2017a; Ingber, 2017b). These codes both run on Classical computers and thereby define a Hybrid Classical-Quantum system that can be calculated solely on a Classical computer.

## 2. Appendix A: Previous Benchmarks

The Ookami SMNI neuroscience benchmarks are fully in the context of the proposed finance project, since similar codes, e.g., ASA and [q]PATHINT are to be used here as well.

On the author's Ookami Testbed account benchmarks were performed on two codes using all 4 compilers available, with and without vectorization. The optimization code Adaptive Simulated Annealing (ASA) has hooks for OpenMP. 100000 iterations/cost functions are required for a complete run.

Each time was obtained from 'time make run' after using 'make compile'. Below are the time estimates. Based on the data below, time for each full run is obtained as 100000 * (0.025 + 10 * 0.07) = 72500s or about 20 hrs/run for 1 subject for one set of data.

Each full run uses 12 subjects, with 2 runs per subject (Train/Test, and switched) = 24 array entries per run. On TACC, multiple instances of a serial job on a single node are run, as described on https://portal.tacc.utexas.edu/software/launcher .

100 such runs for debugging and production would give an estimate required of 48000 hrs. If required, the author's ASA code can limit each run to less than any number of hrs, saving all data required for continued runs.

project path_arm_noVec
12.170u 0.114s 0:12.56 97.7% 0+0k 0+56960io 0pf+0w
path_arm_Vec
12.198u 0.069s 0:13.27 92.3% 0+0k 0+57088io 0pf+0w
path_cc_noVec
9.224u 0.122s 0:09.50 98.3% 0+0k 0+56960io 0pf+0w
path_cc_Vec
9.164u 0.135s 0:09.48 97.9% 0+0k 768+56960io 0pf+0w
path_fcc_noVec
9.251u 0.097s 0:10.13 92.2% 0+0k 0+56960io 0pf+0w
path_fcc_Vec
9.265u 0.067s 0:09.52 97.8% 0+0k 2432+56960io 10pf+0w
path_gcc_noVec
8.740u 0.107s 0:08.94 98.8% 0+0k 0+57088io 0pf+0w
path_gcc_Vec
8.858u 0.067s 0:09.18 97.0% 0+0k 128+56960io 0pf+0w
smni_arm_noVec
34.161u 0.495s 0:35.54 97.4% 0+0k 0+33408io 0pf+0w
smni_arm_Vec
34.066u 0.576s 0:36.13 95.8% 0+0k 128+33152io 0pf+0w

smni_arm_Vec_openmp
5.213u 0.242s 0:05.57 97.8% 0+0k 0+27264io 0pf+0w
smni_cc_noVec
37.009u 0.599s 0:37.87 99.2% 0+0k 0+33408io 0pf+0w
smni_cc_Vec
36.899u 0.568s 0:39.12 95.7% 0+0k 128+33408io 0pf+0w
smni_cc_Vec_openmp
5.157u 0.266s 0:05.49 98.5% 0+0k 768+27264io 0pf+0w
smni_fcc_noVec
33.946u 0.590s 0:35.80 96.4% 0+0k 0+33280io 0pf+0w
smni_fcc_Vec
33.694u 0.730s 0:37.10 92.7% 0+0k 0+33152io 0pf+0w
smni_fcc_Vec_openmp
5.059u 0.282s 0:05.39 98.8% 0+0k 0+27264io 0pf+0w
smni_gcc_noVec
33.006u 0.403s 0:36.88 90.5% 0+0k 0+32896io 0pf+0w
smni_gcc_Vec
33.054u 0.447s 0:34.18 97.9% 0+0k 11520+33152io 0pf+0w
smni_gcc_Vec_openmp
5.351u 0.243s 0:05.90 94.7% 0+0k 256+27520io 0pf+0w

The results sorted on secs yield
path_fcc_Vec 0.067s
path_gcc_Vec 0.067s
path_arm_Vec 0.069s
smni_arm_Vec_openmp 0.242s
smni_gcc_Vec_openmp 0.243s
smni_cc_Vec_openmp 0.266s
smni_fcc_Vec_openmp 0.282

GCC performed very well and was used thereafter.

## 3. Appendix B: Quantum Money

It is reasonable to assume that soon there will be financial derivatives developed on all products. qPATHTREE and qPATHINT can calculate financial derivatives in these spaces. This is beyond using Quantum computation of Classical financial derivatives, since the space of the dependent variables themselves may live in Quantum worlds (Baaquie *et al*, 2002; Piotrowski *et al*, 2005; Accardi & Boukas, 2007; Meyer, 2009; Aaronson & Christiano, 2012; Jogenfors, 2016).

### 3.1. Previous Applications — PATHINT

Options $V$ are generally described by a portfolio $\Pi$ over an underlying asset $S$. The real-world probability distribution of $S$ often is critical to numerical details for trading. The asset is often hedged by trading the option $V$ and a quantity $\Delta$ of the asset $S$.

$$d\Pi = \sigma\left(\frac{\partial V}{\partial S} - \Delta\right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S\right) dt$$

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}\ ,\ \Theta = \frac{\partial \Pi}{\partial t}\ ,\ \Upsilon = \frac{\partial \Pi}{\partial \sigma}\ ,\ \rho = \frac{\partial \Pi}{\partial r} \tag{1}$$

This defines the "Greeks" used in trading. The portfolio $\Pi$ to be hedged is often considered to be "risk-neutral," if $\Delta$ is chosen such that $\Delta = \frac{\partial V}{\partial S}$.

Quite a few closed-form solutions exist for European options (Black & Scholes, 1973) where there is not early exercise. For American options — among the most popular options — there is no general closed

form, and numerical calculations must be performed (Hull, 2000): First the probability "tree" for $S$ is propagated forward in time until the expiration date $T$, branching out as extended $S$ values develop; then, marching back in time, at each time-node calculations of the Greeks above are developed, inserting changes in dividends, interest rates, changes in cheapest-to-deliver for baskets of bonds for options on bond futures, etc (Ingber & Wilson, 1999; Ingber, 2000).

At each node a calculation is performed, comparing the strike price $X$ to the price $S$ at that node, and a decision is made, e.g., whether to exercise the option at that node, which determines the fair value of the option price $V$. To obtain the Greeks, extended derivatives of these derivatives are calculated numerically by using values across branches and notes (Ingber, 2000).

### 3.2. Volatility of Volatility of American Options

An example of a two-dimensional options model processed by PATHINT developed the volatility of volatility of options on old (circa 2000) Eurodollars, using 2-factor model developed by the author (Ingber, 2000):

$$dS = \mu\, S\, dt + \sigma\, F(S, S_0, S_\infty, x, y)\, dz_S$$

$$d\sigma = \nu\, dt + \varepsilon\, dz_\sigma$$

$$F(S, S_0, S_\infty, x, y) = S, \;\; S < S_0$$

$$F(S, S_0, S_\infty, x, y) = S^x S_0^{1-x}, \;\; S_0 \le S \le S_\infty$$

$$F(S, S_0, S_\infty, x, y) = S^y S_0^{1-x} S_\infty^{x-y}, \;\; S > S_\infty \tag{2}$$

where $S_0$ and $S_\infty$ are selected to lie outside the data region used to fit the other parameters, e.g., $S_0 = 1/2$ and $S_\infty = 20$ for fits to Eurodollar futures.

### 3.3. SMFM Example of 2-Factor qPATHINT

qPATHINT was used to perform the similar calculations as were previously performed by PATHINT (Ingber, 2000), except that whereas PATHINT propagates the probability, qPATHINT propagates the wave-function $\psi$ and the kernels are normalized with $\psi^* \psi$. Since real variables were input, the output is the same for test sections of the code as previously reported using one-dimensional closed-form Black-Scholes (Black & Scholes, 1973) and numerical Cox-Ross-Rubenstein (CRR) codes (Cox *et al*, 1979). The output differs for the 2-dimensional volatility of volatility calculations, as $\sigma$ requires averaging. This was another test for qPATHINT, illustrating that if complex variables were input (including real and imaginary non-zero values), then this can be processed by this code.

### 3.4. qPATHINT: Inclusion of Quantum Scales

PATHINT was developed to numerically calculate the path integral, especially for serial changes in time, usually not approachable with standard Monte Carlo techniques.

qPATHINT was developed from the PATHINT C code of about 7500 lines of code. The C code run with the GCC C-compiler uses double complex variables instead of double variables. The code is written for arbitrary N dimensions. The outline of the code is described here for Classical or Quantum systems, using generic coordinates $q$ and $x$ (Ingber, 2016; Ingber, 2017a; Ingber, 2017b):

This histogram procedure recognizes that the distribution (probabilities for Classical systems, wave-functions for Quantum systems) can be numerically approximated by sums of rectangles of height $P_i$ and width $\Delta q^i$ at points $q^i$.

### 3.5. Shocks

Many real-world systems propagate in the presence of continual "shocks" (Ingber, 2017b; Ingber, 2017a):

future dividends
changes in interest rates

changes in asset distributions used in American options algorithms

### 3.6. PATHINT/qPATHINT Histograms

In the prepoint Ito discretization, using a one-dimensional system in variable $x$, the path-integral is written in terms of the kernel $G$, for each of its intermediate integrals, as

$$P(x;t+\Delta t) = \int dx'[g^{1/2}(2\pi\Delta t)^{-1/2}\exp(-L\,\Delta t)]P(x';t) = \int dx' G(x,x';\Delta t)P(x';t)$$

$$P(x;t) = \sum_{i=1}^{N} \pi(x-x^i)P_i(t)$$

$$\pi(x-x^i) = 1\,,\ (x^i - \frac{1}{2}\,\Delta x^{i-1}) \le x \le (x^i + \frac{1}{2}\,\Delta x^i)\ ;\ 0\,,\ \text{otherwise} \tag{3}$$

This yields

$$P_i(t+\Delta t) = T_{ij}(\Delta t)P_j(t)$$

$$T_{ij}(\Delta t) = \frac{2}{\Delta x^{i-1} + \Delta x^i} \int_{x^i-\Delta x^{i-1}/2}^{x^i+\Delta x^i/2} dx \int_{x^j-\Delta x^{j-1}/2}^{x^j+\Delta x^j/2} dx' G(x,x';\Delta t) \tag{4}$$

$T_{ij}$ is a banded matrix representing the Gaussian nature of the short-time probability centered about the (possibly time-dependent) drift.

Several projects have used this algorithm (Wehner & Wolfer, 1983a; Wehner & Wolfer, 1983b; Wehner & Wolfer, 1987; Ingber & Nunez, 1995; Ingber, Srinivasan & Nunez, 1996; Ingber & Wilson, 1999). 2-dimensional codes were developed for Statistical Mechanics of Combat (SMC), Statistical Mechanics of Neocortical Interactions (SMNI) and Statistical Mechanics of Financial Markets (SMFM) (Ingber, Fujio & Wehner, 1991; Ingber & Nunez, 1995; Ingber, 2000).

### 3.7. Meshes For [q]PATHINT

Explicit dependence of $L$ on time $t$ also can be included. The mesh $\Delta q^i$ is strongly dependent on diagonal elements of the diffusion matrix, e.g.,

$$\Delta q^i \approx (\Delta t g^{|ii|})^{1/2} \tag{5}$$

The covariances develop a rectangular-ish underlying mesh. Fitting the data with integrals over the short-time distribution permits the coarser meshes than the corresponding stochastic differential equation(s) (Ingber, 1990; Wehner & Wolfer, 1983a). Here, vertical bars on $|ii|$ imply no summation.

### 3.8. Broad-Banded Kernels Required

SMNI qPATHINT requires broad-banded kernels for oscillatory Quantum states.

SMFM PATHTREE and qPATHTREE are different options codes based on path-integral studies permitting a new very fast binary calculation applied to nonlinear time-dependent systems (Ingber, Chen *et al*, 2001). However, currently an SMFM [q]PATHTREE is only a binary tree with J = 1 and it cannot be effectively applied to Quantum oscillatory systems (Ingber, 2016; Ingber, 2017a; Ingber, 2017b).

### 3.9. SMFM qPATHINT With Serial Shocks

Some N-dim qPATHINT runs for SMFM used a contrived N-factor model with the same 1-dimensional system cloned in all dimensions (each unit is a "double complex"). The kernel size is $(\mathrm{I\,J})^{\mathrm{N}}$, where I = imxall, J = jmxall (= kernel band width), and kernel size = ijkcnt. This spatial mesh might change at each time slice.

D=1:imxall: 27 , jmxall: 7 , ijkcnt: 189
D=2:imxall: 729 , jmxall: 49 , ijkcnt: 35721

D=3:imxall: 19683 , jmxall: 343 , ijkcnt: 6751269
D=4:imxall: 531441 , jmxall: 2401 , ijkcnt: 1275989841
D=5:imxall: 14348907 , jmxall: 16807 , ijkcnt: 241162079949
D=6:imxall: 387420489 , jmxall: 117649 , ijkcnt: 45579633110361
D=7:imxall: 10460353203 , jmxall: 823543 , ijkcnt: 8614550657858229

A full set of Adaptive Simulated Annealing (ASA) (Ingber, 1993a) fits of Classical SMNI to EEG data takes about 1000 hours of supercomputer CPUs; many such sets of runs are required to develop good models. Cost functions including Quantum processes take even longer.

### 3.10. Applications

#### 3.10.1. Quantum Money and Blockchains

Soon financial derivatives will be developed on these products. qPATHINT can calculate financial derivatives in these spaces. The marketplace will determine traded variables.

#### 3.10.2. Enhanced Security/Verification

Quantum currency cannot be cloned. Such currencies are candidates for efficient blockchains. Each "coin" has a unique identity (Meyer, 2009; Aaronson & Christiano, 2012; Bartkiewicz *et al*, 2016; Jogenfors, 2016). There are issues about the decoherence time of such currency that are being addressed, e.g., by using toplogical qubits (Quantum bits).

## 4. Appendix C: Path Integrals

### 4.1. Path Integral in Stratonovich (Midpoint) Representation

Most suitable for examining discretization issues in time-dependent nonlinear systems, the path integral in the Feynman (midpoint) representation is (Langouche *et al*, 1979; Schulman, 1981; Langouche *et al*, 1982) (N.b. $g^{\dagger}$ in $DM$ implies a prepoint evaluation.) Unless stated otherwise, the Einstein summation convention is used for repeated indices to signify summation.

$$P[M_t|M_{t_0}]dM(t) = \int\cdots\int DM \exp\left(-\min\int_{t_0}^{t} dt' L\right)\delta(M(t_0)=M_0)\;\delta(M(t)=M_t)$$

$$DM = \lim_{u\to\infty}\prod_{\rho=1}^{u+1} g^{\dagger 1/2}\prod_{G}(2\pi\theta)^{-1/2}dM_\rho^G$$

$$L(\dot{M}^G, M^G, t) = \frac{1}{2}(\dot{M}^G - h^G)g_{GG'}(\dot{M}^{G'} - h^{G'}) + \frac{1}{2}h^G{}_{;G} + R/6 - V$$

$$\dot{M}^G(t)\to M^G_{\rho+1} - M^G_\rho\;,\; M^G(t)\to\frac{1}{2}(M^G_{\rho+1} + M^G_\rho)\;,\; [\cdots]_{,G} = \frac{\partial[\cdots]}{\partial M^G}$$

$$h^G = g^G - \frac{1}{2}g^{-1/2}(g^{1/2}g^{GG'})_{,G'}\;,\; h^G{}_{;G} = h^G_{,G} + \Gamma^F_{GF}h^G = g^{-1/2}(g^{1/2}h^G)_{,G}$$

$$g_{GG'} = (g^{GG'})^{-1}\;,\; g = \det(g_{GG'})$$

$$\Gamma^F_{JK} \equiv g^{LF}[JK,L] = g^{LF}(g_{JL,K} + g_{KL,J} - g_{JK,L})$$

$$R = g^{JL}R_{JL} = g^{JL}g^{JK}R_{FJKL}$$

$$R_{FJKL} = \frac{1}{2}(g_{FK,JL} - g_{JK,FL} - g_{FL,JK} + g_{JL,FK}) + g_{MN}(\Gamma^M_{FK}\Gamma^N_{JL} - \Gamma^M_{FL}\Gamma^N_{JK}) \tag{6}$$

### 4.2. Path Integral in Ito (Prepoint) Representation

For probability distributions or for wave functions the Ito development in the prepoint representation:

$$P[M_t|M_{t_0}]dM(t) = \int \cdots \int DM \exp(-\min \int_{t_0}^{t} dt' L)\delta(M(t_0) = M_0)\delta(M(t) = M_t)$$

$$DM = \lim_{u\to\infty} \prod_{\rho=1}^{u+1} g^{1/2} \prod_G (2\pi\Delta t)^{-1/2} dM_\rho^G$$

$$L(\dot{M}^G, M^G, t) = \frac{1}{2}(\dot{M}^G - g^G)g_{GG'}(\dot{M}^{G'} - g^{G'}) + R/6$$

$$\dot{M}^G(t) \to M_{\rho+1}^G - M_\rho^G \ , \ M^G(t) \to M_\rho^G$$

$$g_{GG'} = (g^{GG'})^{-1} \ , \ g = \det(g_{GG'}) \tag{7}$$

The diagonal diffusion terms are $g^{|GG|}$ and the drift terms are $g^G$; Here, vertical bars on $|GG|$ imply no summation. No-constant diffusions terms add terms to the drifts. A Riemannian-curvature potential $R/6$ arises for dimension > 1 in the midpoint Stratonovich/Feynman discretization for nonlinear systems.

### 4.3. Path-Integral Riemannian Geometry

The midpoint derivation derives a Riemannian geometry with a metric defined by the inverse of the covariance matrix

$$g_{GG'} = (g^{GG'})^{-1} \tag{8}$$

and where $R$ is the Riemannian curvature

$$R = g^{JL}R_{JL} = g^{JL}g^{JK}R_{FJKL} \tag{9}$$

The Ito prepoint discretization gives a simpler algebraic form,

$$M(\bar{t}_s) = M(t_s)$$

$$\underline{L} = \frac{1}{2}(dM^G/dt - g^G)g_{GG'}(dM^{G'}/dt - g^{G'}) - V \tag{10}$$

but the Lagrangian $\underline{L}$ is useful only in the weak-noise limit. This means that finer meshes are often required for the prepoint case.

### 4.4. Three Approaches Are Mathematically Equivalent

Three basic different approaches are mathematically equivalent:

(a) Fokker-Planck/Chapman-Kolmogorov partial-differential equations
(b) Langevin coupled stochastic-differential equations
(c) Lagrangian or Hamiltonian path-integrals, as defined above

The path-integral approach is particularly useful to precisely define intuitive physical variables from the Lagrangian $L$ in terms of its underlying variables $M^G$:

$$\text{Momentum: } \Pi^G = \frac{\partial L}{\partial(\partial M^G/\partial t)}$$

$$\text{Mass: } g_{GG'} = \frac{\partial L}{\partial(\partial M^G/\partial t)\partial(\partial M^{G'}/\partial t)}$$

$$\text{Force: } \frac{\partial L}{\partial M^G}$$

$$F = ma: \ \delta L = 0 = \frac{\partial L}{\partial M^G} - \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial M^G / \partial t)} \tag{11}$$

The Laplacian also can be considered as an "average" over the Lagrangian (Ingber, 2022a).

Differentiation of noisy systems can introduce more noise, whereas integration is inherently a smoothing process and the path-integral often gives superior numerical performance to other representations below.

#### 4.4.1. Stochastic Differential Equation (SDE)

The Stratonovich (midpoint discretized) Langevin equations can be analyzed in terms of the Wiener process $dW^i$, which can be rewritten in terms of Gaussian noise $\eta^i = dW^i/dt$ if care is taken in the limit.

$$dM^G = f^G(t, M(t))dt + \hat{g}_i^G(t, M(t))dW^i$$

$$\dot{M}^G(t) = f^G(t, M(t)) + \hat{g}_i^G(t, M(t))\eta^i(t)$$

$$dW^i \to \eta^i dt$$

$$M = \{ M^G; G = 1, \cdots, \Lambda \}$$

$$\eta = \{ \eta^i; i = 1, \cdots, N \}$$

$$\dot{M}^G = dM^G/dt$$

$$< \eta^j(t) >_\eta = 0 \ , < \eta^j(t), \eta^{j'}(t') >_\eta = \delta^{jj'} \delta(t - t') \tag{12}$$

$\eta^i$ represents Gaussian white noise.

#### 4.4.2. Partial Differential Equation (PDE)

The Fokker-Planck, or Chapman-Kolmogorov, partial differential equation is:

$$P_{,t} = \frac{1}{2} (g^{GG'} P)_{,GG'} - (g^G P)_{,G} + VP$$

$$P = < P_\eta >_\eta$$

$$g^G = f^G + \frac{1}{2} \hat{g}_i^{G'} \hat{g}_{i,G'}^G$$

$$g^{GG'} = \hat{g}_i^G \hat{g}_i^{G'}$$

$$(\cdots)_{,G} = \partial(\cdots)/\partial M^G \tag{13}$$

$g^G$ replaces $f^G$ in the SDE if the Ito (prepoint discretized) calculus is used to define that equation. If boundary conditions are added as Lagrange multipliers, these enter as a "potential" $V$, creating a Schrodinger-type equation.

### 4.5. Applications

Path integrals and PATHINT have been applied across several disciplines, including combat simulations (Ingber, Fujio & Wehner, 1991), neuroscience (Ingber, 1994; Ingber & Nunez, 1995; Ingber & Nunez, 2010; Ingber, 2017c; Ingber, 2018), finance (Ingber & Wilson, 2000; Ingber, 2000; Ingber, Chen *et al*, 2001; Ingber, 2016; Ingber, 2017a; Ingber, 2017b; Ingber, 2017c), and other nonlinear systems (Ingber, 1995; Ingber, Srinivasan & Nunez, 1996; Ingber, 1998a).

## 5. Appendix D: Adaptive Simulated Annealing (ASA)

### 5.1. Importance Sampling

Nonlinear and/or stochastic systems often require importance-sampling algorithms to scan or to fit parameters. Methods of simulated annealing (SA) are often used. Proper annealing (not "quenching") possesses a proof of finding the deepest minimum in searches.

The ASA code is open-source software, and can be downloaded and used without any cost or registration at https://www.ingber.com/#ASA (Ingber, 1993a; Ingber, 2012a).

This algorithm fits empirical data to a cost function over a $D$-dimensional parameter space, adapting for varying sensitivities of parameters during the fit.

This ASA algorithm, with a true exponential annealing schedule, is faster than fast Cauchy annealing, which has schedule $T_i = T_0/k$, and much faster than Boltzmann annealing, which has schedule $T_i = T_0/\ln k$ (Ingber, 1989).

### 5.2. Outline of ASA Algorithm

For parameters

$$\alpha_k^i \in [A_i, B_i]$$

sampling with the random variable $x^i$

$$x^i \in [-1, 1]$$

$$\alpha_{k+1}^i = \alpha_k^i + x^i (B_i - A_i)$$

the default generating function is

$$g_T(x) = \prod_{i=1}^{D} \frac{1}{2\ln(1 + 1/T_i)(|x^i| + T_i)} \equiv \prod_{i=1}^{D} g_T^i(x^i)$$

in terms of parameter "temperatures"

$$T_i = T_{i0} \exp(-c_i k^{1/D}) \tag{14}$$

The default ASA uses the same type of annealing schedule for the acceptance function $h$ as used for the generating function $g$.

All default functions in ASA can be overridden with user-defined functions.

### 5.3. ASA Applications

The ASA code (Ingber, 1993a) and the original Very Fast Simulated Reannealing (VFSR) code (Ingber, 1989) have been used by many researchers, including the author, in several disciplines:

chaotic systems (Ingber, Srinivasan & Nunez, 1996)
combat simulations (Ingber, 1993c; Ingber, 1998a)
financial systems [bonds, equities, futures, options] (Ingber, 1990; Ingber, 1996b; Ingber, 2000; Ingber, Chen *et al*, 2001; Ingber & Mondescu, 2003; Ingber, 2005)
neuroscience (Ingber, 1991; Ingber, 1992; Ingber & Nunez, 1995; Ingber, Srinivasan & Nunez, 1996; Ingber, 1996c; Ingber, 1997; Ingber, 1998b; Ingber, 2006; Ingber, 2009b; Ingber, 2009a; Ingber & Nunez, 2010; Ingber, 2012b; Ingber, 2012c; Nunez *et al*, 2013; Ingber, 2013; Ingber, Pappalepore & Stesiak, 2014; Ingber, 2015)
optimization (Ingber, 1989; Ingber & Rosen, 1992; Ingber, 1993b; Ingber, 1996a; Atiya *et al*, 2003; Ingber, 2012a)

**Acknowledgements**

The author thanks both XSEDE.org (now ACCESS-CI.org) for supercomputer resources from Feb 2013 through 2021, and the Ookami supercomputer at StonyBrook.edu from Jun 2021 through May 2023.

**REFERENCES**

S. Aaronson & P. Christiano (2012). Quantum money from hidden subspaces. arXiv:1203.4740 [quant-ph]. MIT.

L. Accardi & A. Boukas (2007). The quantum Black-Scholes equation. arXiv:0706.1300 [q-fin.PR]. U di Roma Torvergata.

A.F. Atiya, A.G. Parlos & L. Ingber (2003). A reinforcement learning method based on adaptive simulated annealing, In: Proceedings International Midwest Symposium on Circuits and Systems (MWCAS), December 2003, IEEE CAS, 121-124. [URL https://www.ingber.com/asa03_reinforce.pdf ]

B.E. Baaquie, C. Coriano & M. Srikant (2002). Quantum mechanics, path integrals and option pricing: Reducing the complexity of finance. arXiv:cond-mat/0208191 [cond-mat.soft]. National U Singapore.

K. Bartkiewicz, A. Cernoch, G. Chimczak, K. Lemr, A. Miranowicz & F. Nori (2016). Experimental quantum forgery of quantum optical money. arXiv:1604.04453v1 [quant-ph]. Adam Mickiewicz University.

M. Benedetti, E. Lloyd, S. Sack & M. Fiorentini (2019). Parameterized quantum circuits as machine learning models. Quantum Science and Technology. 4(4). [URL https://doi.org/10.1088/2058-9565/ab5944 ]

F. Black & M. Scholes (1973). The pricing of options and corporate liabilities. The Journal of Political Economy. 81(3), 637-659.

S. Chakrabarti, R. Krishnakumar, G. Mazzola, N. Stamatopoulos, S. Woerner & W.J. Zeng (2020). A threshold for quantum advantage in derivative pricing. arXiv:2012.03819v2 [quant-ph]. Goldman Sachs.

J.C. Cox, S.A. Ross & M. Rubenstein (1979). Option pricing: A simplified approach. Journal Financial Economics. 7, 229-263.

J.C. Hull (2000). Options, Futures, and Other Derivatives, 4th Edition. Prentice Hall, Upper Saddle River, NJ.

L. Ingber (1989). Very fast simulated re-annealing. Mathematical Computer Modelling. 12(8), 967-973. [URL https://www.ingber.com/asa89_vfsr.pdf ]

L. Ingber (1990). Statistical mechanical aids to calculating term structure models. Physical Review A. 42(12), 7057-7064. [URL https://www.ingber.com/markets90_interest.pdf ]

L. Ingber (1991). Statistical mechanics of neocortical interactions: A scaling paradigm applied to electroencephalography. Physical Review A. 44(6), 4017-4060. [URL https://www.ingber.com/smni91_eeg.pdf ]

L. Ingber (1992). Generic mesoscopic neural networks based on statistical mechanics of neocortical interactions. Physical Review A. 45(4), R2183-R2186. [URL https://www.ingber.com/smni92_mnn.pdf ]

L. Ingber (1993a). Adaptive Simulated Annealing (ASA). Global optimization C-code. Caltech Alumni Association. [URL https://www.ingber.com/#ASA-CODE ]

L. Ingber (1993b). Simulated annealing: Practice versus theory. Mathematical Computer Modelling. 18(11), 29-57. [URL https://www.ingber.com/asa93_sapvt.pdf ]

L. Ingber (1993c). Statistical mechanics of combat and extensions, In: Toward a Science of Command, Control, and Communications, ed. C. Jones. American Institute of Aeronautics and Astronautics, 117-149. [ISBN 1-56347-068-3. URL https://www.ingber.com/combat93_c3sci.pdf ]

L. Ingber (1994). Statistical mechanics of neocortical interactions: Path-integral evolution of short-term memory. Physical Review E. 49(5B), 4652-4664. [URL https://www.ingber.com/smni94_stm.pdf ]

L. Ingber (1995). Path-integral evolution of multivariate systems with moderate noise. Physical Review E. 51(2), 1616-1619. [URL https://www.ingber.com/path95_nonl.pdf ]

L. Ingber (1996a). Adaptive simulated annealing (ASA): Lessons learned. Control and Cybernetics. 25(1), 33-54. [Invited paper to Control and Cybernetics on Simulated Annealing Applied to Combinatorial Optimization. URL https://www.ingber.com/asa96_lessons.pdf ]

L. Ingber (1996b). Canonical momenta indicators of financial markets and neocortical EEG, In: Progress in Neural Information Processing, ed. S.-I. Amari, L. Xu, I. King & K.-S. Leung. Springer, 777-784. [Invited paper to the 1996 International Conference on Neural Information Processing (ICONIP'96), Hong Kong, 24-27 September 1996. ISBN 981-3083-05-0. URL https://www.ingber.com/markets96_momenta.pdf ]

L. Ingber (1996c). Statistical mechanics of neocortical interactions: Multiple scales of EEG, In: Frontier Science in EEG: Continuous Waveform Analysis (Electroencephal. clin. Neurophysiol. Suppl. 45), ed. R.M. Dasheiff & D.J. Vincent. Elsevier, 79-112. [Invited talk to Frontier Science in EEG Symposium, New Orleans, 9 Oct 1993. ISBN 0-444-82429-4. URL https://www.ingber.com/smni96_eeg.pdf ]

L. Ingber (1997). Statistical mechanics of neocortical interactions: Applications of canonical momenta indicators to electroencephalography. Physical Review E. 55(4), 4578-4593. [URL https://www.ingber.com/smni97_cmi.pdf ]

L. Ingber (1998a). Data mining and knowledge discovery via statistical mechanics in nonlinear stochastic systems. Mathematical Computer Modelling. 27(3), 9-31. [URL https://www.ingber.com/path98_datamining.pdf ]

L. Ingber (1998b). Statistical mechanics of neocortical interactions: Training and testing canonical momenta indicators of EEG. Mathematical Computer Modelling. 27(3), 33-64. [URL https://www.ingber.com/smni98_cmi_test.pdf ]

L. Ingber (2000). High-resolution path-integral development of financial options. Physica A. 283(3-4), 529-558. [URL https://www.ingber.com/markets00_highres.pdf ]

L. Ingber (2005). Trading in Risk Dimensions (TRD). Report 2005:TRD. Physical Studies Institute LLC. [URL https://www.ingber.com/markets05_trd.pdf ]

L. Ingber (2006). Statistical mechanics of neocortical interactions: Portfolio of physiological indicators. Report 2006:PPI. Physical Studies Institute LLC. [URL https://www.ingber.com/smni06_ppi.pdf ]

L. Ingber (2009a). Statistical mechanics of neocortical interactions: Nonlinear columnar electroencephalography. NeuroQuantology Journal. 7(4), 500-529. [URL https://www.ingber.com/smni09_nonlin_column_eeg.pdf ]

L. Ingber (2009b). Statistical mechanics of neocortical interactions: Portfolio of physiological indicators. The Open Cybernetics Systemics Journal. 3(14), 13-26. [URL https://doi.org/10.2174/1874110x00903010013 ]

L. Ingber (2012a). Adaptive Simulated Annealing, In: Stochastic global optimization and its applications with fuzzy adaptive simulated annealing, ed. H.A. Oliveira, Jr., A. Petraglia, L. Ingber, M.A.S. Machado & M.R. Petraglia. Springer, 33-61. [Invited Paper. URL https://www.ingber.com/asa11_options.pdf ]

L. Ingber (2012b). Columnar EEG magnetic influences on molecular development of short-term memory, In: Short-Term Memory: New Research, ed. G. Kalivas & S.F. Petralia. Nova, 37-72. [Invited Paper. URL https://www.ingber.com/smni11_stm_scales.pdf ]

L. Ingber (2012c). Influence of macrocolumnar EEG on Ca waves. Current Progress Journal. 1(1), 4-8. [URL https://www.ingber.com/smni12_vectpot.pdf ]

L. Ingber (2013). Electroencephalographic (EEG) influence on Ca2+ waves: Lecture plates. Report 2013:LEFI. Physical Studies Institute LLC. [2nd World Neuroscience Online Conference 18 June 2013. URL https://www.ingber.com/smni13_eeg_ca_lect.pptx and https://www.ingber.com/smni13_eeg_ca_lect.pdf ]

L. Ingber (2015). Calculating consciousness correlates at multiple scales of neocortical interactions, In: Horizons in Neuroscience Research, ed. A. Costa & E. Villalba. Nova, 153-186. [ISBN 978-1-63482-632-7. Invited paper. URL https://www.ingber.com/smni15_calc_conscious.pdf ]

L. Ingber (2016). Path-integral quantum PATHTREE and PATHINT algorithms. International Journal of Innovative Research in Information Security. 3(5), 1-15. [URL https://www.ingber.com/path16_quantum_path.pdf ]

L. Ingber (2017a). Evolution of regenerative Ca-ion wave-packet in neuronal-firing fields: Quantum path-integral with serial shocks. International Journal of Innovative Research in Information Security. 4(2), 14-22. [URL https://www.ingber.com/path17_quantum_pathint_shocks.pdf ]

L. Ingber (2017b). Options on quantum money: Quantum path-integral with serial shocks. International Journal of Innovative Research in Information Security. 4(2), 7-13. [URL https://www.ingber.com/path17_quantum_options_shocks.pdf ]

L. Ingber (2017c). Quantum Path-Integral qPATHINT Algorithm. The Open Cybernetics Systemics Journal. 11, 119-133. [URL https://www.ingber.com/path17_qpathint.pdf and https://doi.org/10.2174/1874110X01711010119 ]

L. Ingber (2018). Quantum calcium-ion interactions with EEG. Sci. 1(7), 1-21. [URL https://www.ingber.com/smni18_quantumCaEEG.pdf and https://doi.org/10.3390/sci1010020 ]

L. Ingber (2019). Quantum-Classical interactions: calcium ions and synchronous neural firings. Acta Scientific Computer Sciences. 2(1), 13-20. [URL https://www.ingber.com/smni19_quantum-classical.pdf and https://www.actascientific.com/ASCS/pdf/quantum-classical-interactions-calcium-ions-and-synchronous-neural-firings.pdf ]

L. Ingber (2020). Developing bid-ask probabilities for high-frequency trading. Virtual Economics. 3(2), 7-24. [URL https://www.ingber.com/markets19_bid_ask_prob.pdf and https://doi.org/10.34021/ve.2020.03.02(1) ]

L. Ingber (2021a). Hybrid classical-quantum computing: Applications to statistical mechanics of financial markets. E3S Web Conference. 307, 04001-04012. [URL https://www.ingber.com/markets21_hybrid_smfm.pdf and https://doi.org/10.1051/e3sconf/202130704001 ]

L. Ingber (2021b). Hybrid classical-quantum computing: Applications to statistical mechanics of neocortical interactions. Global Journal Human-Social Science. 21(6), 1-12. [URL https://www.ingber.com/smni21_hybrid_smni.pdf ]

L. Ingber (2021c). Revisiting our quantum world, In: Technological Breakthroughs and Future Business Opportunities in Education, Health, and Outer Space, ed. A. Hooke. IGI Global, 1-16. [URL https://www.ingber.com/quantum20_techcast.pdf ]

L. Ingber (2022a). From Lagrangian To Laplacian: An Example From EEG Standing Waves. Report 2022:LLESW. Physical Studies Institute LLC. [URL https://www.ingber.com/smni22_Lagrangian-Laplacian.pdf ]

L. Ingber (2022b). Quantum calcium-ion affective influences measured by EEG. Journal of Research in Applied Mathematics. 8(1), 15-21. [URL https://www.ingber.com/quantum20_affective.pdf ]

L. Ingber (2022c). Realistic Neural Networks. American Journal of Engineering Research. 11(11), 8-10. [URL https://www.ingber.com/smni22_realistic_neural_nets.pdf ]

L. Ingber, C. Chen, R.P. Mondescu, D. Muzzall & M. Renedo (2001). Probability tree algorithm for general diffusion processes. Physical Review E. 64(5), 056702-056707. [URL https://www.ingber.com/path01_pathtree.pdf ]

L. Ingber, H. Fujio & M.F. Wehner (1991). Mathematical comparison of combat computer models to exercise data. Mathematical Computer Modelling. 15(1), 65-90. [URL https://www.ingber.com/combat91_data.pdf ]

L. Ingber & R.P. Mondescu (2003). Automated internet trading based on optimized physics models of markets, In: Intelligent Internet-Based Information Processing Systems, ed. R.J. Howlett, N.S. Ichalkaranje, L.C. Jain & G. Tonfoni. World Scientific, 305-356. [Invited paper. URL

https://www.ingber.com/markets03_automated.pdf ]

L. Ingber & P.L. Nunez (1995). Statistical mechanics of neocortical interactions: High resolution path-integral calculation of short-term memory. Physical Review E. 51(5), 5074-5083. [URL https://www.ingber.com/smni95_stm.pdf ]

L. Ingber & P.L. Nunez (2010). Neocortical Dynamics at Multiple Scales: EEG Standing Waves, Statistical Mechanics, and Physical Analogs. Mathematical Biosciences. 229, 160-173. [URL https://www.ingber.com/smni10_multiple_scales.pdf and https://doi.org/10.1016/j.mbs.2010.12.003 ]

L. Ingber, M. Pappalepore & R.R. Stesiak (2014). Electroencephalographic field influence on calcium momentum waves. Journal of Theoretical Biology. 343, 138-153. [URL https://www.ingber.com/smni14_eeg_ca.pdf and https://doi.org/10.1016/j.jtbi.2013.11.002 ]

L. Ingber & B. Rosen (1992). Genetic algorithms and very fast simulated reannealing: A comparison. Mathematical Computer Modelling. 16(11), 87-100. [URL https://www.ingber.com/asa92_saga.pdf ]

L. Ingber, R. Srinivasan & P.L. Nunez (1996). Path-integral evolution of chaos embedded in noise: Duffing neocortical analog. Mathematical Computer Modelling. 23(3), 43-53. [URL https://www.ingber.com/path96_duffing.pdf ]

L. Ingber & J.K. Wilson (1999). Volatility of volatility of financial markets. Mathematical Computer Modelling. 29(5), 39-57. [URL https://www.ingber.com/markets99_vol.pdf ]

L. Ingber & J.K. Wilson (2000). Statistical mechanics of financial markets: Exponential modifications to Black-Scholes. Mathematical Computer Modelling. 31(8/9), 167-192. [URL https://www.ingber.com/markets00_exp.pdf ]

J. Jogenfors (2016). Quantum bitcoin: An anonymous and distributed currency secured by the no-cloning theorem of quantum mechanics. arXiv:1604.01383 [quant-ph]. Linkoping U.

F. Langouche, D. Roekaerts & E. Tirapegui (1979). Discretization problems of functional integrals in phase space. Physical Review D. 20, 419-432.

F. Langouche, D. Roekaerts & E. Tirapegui (1982). Functional Integration and Semiclassical Expansions. Reidel, Dordrecht, The Netherlands.

K. Meyer (2009). Extending and simulating the quantum binomial options pricing model. Thesis. U Manitoba. [URL http://hdl.handle.net/1993/3154 ]

P.L. Nunez, R. Srinivasan & L. Ingber (2013). Theoretical and experimental electrophysiology in human neocortex: Multiscale correlates of conscious experience, In: Multiscale Analysis and Nonlinear Dynamics: From genes to the brain, ed. M.M. Pesenson. Wiley, 149-178. [URL https://doi.org/10.1002/9783527671632.ch06 ]

E.W. Piotrowski, M. Schroeder & A. Zambrzycka (2005). Quantum extension of European option pricing based on the Ornstein-Uhlenbeck process. Physica A. 368(1), 176-182.

L.S. Schulman (1981). Techniques and Applications of Path Integration. J. Wiley & Sons, New York.

M.F. Wehner & W.G. Wolfer (1983a). Numerical evaluation of path-integral solutions to Fokker-Planck equations. I.. Physical Review A. 27, 2663-2670.

M.F. Wehner & W.G. Wolfer (1983b). Numerical evaluation of path-integral solutions to Fokker-Planck equations. II. Restricted stochastic processes. Physical Review A. 28, 3003-3011.

M.F. Wehner & W.G. Wolfer (1987). Numerical evaluation of path integral solutions to Fokker-Planck equations. III. Time and functionally dependent coefficients. Physical Review A. 35, 1795-1801.

N. Zulfiqar & S. Gulzar (2021). Implied volatility estimation of bitcoin options and the stylized facts of option pricing. Financial Innovation. 7(67).

$Id: https://www.ingber.com/markets22_quantum_options.pdf 1.55 2022/12/17 15:58:32 ingber Exp ingber $