UNIVERSITY OF CALIFORNIA

San Diego

One-Meson-Exchange Potentials and Properties of Nucleon-Nucleon Scattering and of Nuclear Matter

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Physics

by

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1967

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PUBLICATIONS

"Non-Adiabatic Corrections to the Method of Stationary States," Phys. Rev. 139, A35-A39 (1965)

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ABSTRACT OF THE DISSERTATION

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Professor Keith A. Brueckner, Chairman

The problem this thesis undertakes is to define an interaction between nucleons to be used to consistently calculate:

1. Nucleon-nucleon elastic scattering processes reproducing experimental scattering data (phase shift analyses).

2. The binding energy and equilibrium density of a 'large' system of nucleons (nuclear matter).

The nature of these two problems require that this interaction:

1. Be applicable throughout a laboratory incident energy range of about 0 - 400 MeV [below inelastic thresholds],

2. be defined with minimum phenomonology - i.e., to use as much theoretic guidance as possible to define a unique interaction.

This last requirement actually prompted this work as phase-shift analyses of nucleon-nucleon scattering seem to be relatively insensitive to various functional forms of potentials, compared to the more sensitive nuclear matter calculations using the same potentials. Since the last major hurdle to understanding of nuclear matter appeared to be the inability to define such a potential, this work was undertaken after the author had listened to an enlightening seminar given by David Wong on the use of a one-meson exchange model to rather successfully calculate nucleon-nucleon scattering processes using dispersion-relation theory. David Wong also suggested a method for deriving a potential to be used in Schroedinger's theory, which is necessary to solve the nuclear matter problem.

The total Langrangian describing nucleons and their interaction via one-meson exchanges was non-relativistically reduced, corresponding to a Schroedinger equation with a velocity-dependent (<u>not</u> energy-dependent) potential of the form described by Scotti-Wong (modified to interpret cutoffs and momentum dependence). To preserve the correct eigenvalue properties of the integral-equation

formulation of the two-body Schroedinger's equation, it was found necessary to keep the potential of operator form (i. e., containing differential operators), and not merely a function of the radial parameter. However, integration by parts allows one to operate on the Green's function instead of on the wave-function. As the Green's function is expressed in simple analytic (differentiable) functions, the resulting integral-equation is really no harder to solve than one involving a simple potential function. Keeping the potential in this operator form also allows one to correctly project the potential from a set of five invariant helicity amplitudes onto the more common, though phenomonological, set of five invariant Wigner forms which are eigen-operators of the partial-wave reduced wave-functions, in turn expressed as eigenfunctions of the total angular momentum.

Then this potential, so 'uniquely' defined, and giving analytic functions throughout coordinate space, is similarly defined to perform a modified nuclear matter calculation similar to that done by Brueckner and Masterson.

When compared with other current potentials, this potential, with only 8 parameters (themselves reasonable physical quantities), proves itself to be the best overall description of the nucleon-nucleon interaction, after doing detailed calculations of nucleon-nucleon scattering, the deuteron state, and nuclear matter.

I. OUTLINE OF FORMALISM

All previous theoretic and experimental studies certainly make it more plausible to state that a force between two nucleons arises from their mutual interaction via mesons rather than from their mutual interaction with any arbitrary function with parameters fitted to experimental data. That nucleons and mesons themselves may be various eigenstates as something more basic, as "energy" itself, is a deeper question and as yet not subject to such refined calculations as will be performed here. The problem to date has been to make the transition from words about meson exchange to an 'exact' formalism to deal with nuclear forces.

In the truly nonrelativistic energy region (where the kinetic energy of the colliding nucleons is less than the rest mass of a nucleon), remarkable progress has been made by Scotti and Wong.[1] They postulated forces due to meson exchange via a Lagrangian interaction, and (essentially) unitarized this interaction Lagrangian via method of dispersion relations. Despite some uncertainties associated with descriptions of the mesons (Regge cutoffs, etc.) and with dispersion theory itself (subtraction, unitarization, etc.), a good fit to the experimental data was achieved within the elastic scattering region (0 - 400 MeV incident lab energy).

However, in this energy region, Schroedinger's theory is surely as exact as the experimental data for the free nucleons, and it

would be most reassuring if an interacting system of nucleons could be described with a formalism consistent with Schroedinger's equation. The method is straightforward: A non-relativistic reduction of a Lagrangian Field Theory describing free nucleons yields the Schroedinger equation upon minimization of the Action. It is now proposed to make a similar non-relativistic reduction of the total Lagrangian (including interactions via meson exchange) to the same 'order' [$\sim k^2/m_n^2$ (momentum/mass)^2] and attempt to 'correctly' solve the resulting equations. It is indeed 'fundamental', that within the bounding assumptions of (one) meson-exchanges being responsible for nuclear forces, the potential derived here is as meaningful as Schroedinger's equation which in turn can be thought of (and possibly someday may be proved) as being derived from a Field Theory, amiable to the calculation of a scattering matrix from the assumption of existence of a Lagrangian. This is one theory which acknowledges the simultaneous achievement of, and challenge to, modern physics in apparently coming within the grasp of a mathematically sound and useful description of our human perceptions of nature via experiment. However, besides lacking absolute mathematical vigor, this is not the only 'acceptable' physical theory [i.e., there exists Hamiltonian formalisms], but for the lack of such a unique theory it seems most feasible to try to approach the problem as consistently as possible.

Also ignored are other deficiencies of the theory, such as those associated with divergences of the ρ vector meson.

Therefore a few words about Lagrangians and fields seems in order. The following brief outline will also serve to illustrate some highlights in the development of scattering and many-body theories. The references given will yield much detail presently known in published form.

If one defines,

σ_0 = arbitrary complex number

$\sigma_1 = \sigma_1(\underset{\sim}{k})$ - arbitrary square integrable function in the 3-dimensional space, R^3, of vectors $\underset{\sim}{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$

$\sigma_j = \sigma_j(\underset{\sim}{k}_1, \ldots, \underset{\sim}{k}_j)$, symmetric or antisymmetric, depending on statistics, in $\underset{\sim}{k}_1, \ldots, \underset{\sim}{k}_j$ square integrable in R^{3j}, (I-1)

then Hilbert space, $\mathcal{H}$, consists of elements:

$$|\sigma> = |\sigma_o, \sigma_1, \ldots, \sigma_j, \ldots, 0> \qquad \text{(I-2)}$$

where all but a finite number of entries are zero, (plus its completion), i.e., $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_j \oplus \ldots$. A "vacuum state" is defined as

$$|0> = |1, 0, \ldots, 0> . \qquad \text{(I-3)}$$

Addition and scalar multiplication is to be taken component-wise and an inner product is defined:

$$\langle \rho \mid \sigma \rangle = \langle \rho_0, \rho_1, \ldots \mid \sigma_0, \sigma_1, \ldots \rangle$$

$$= \bar{\rho}_0 \sigma_0 + \sum_{j=1}^{\infty} \int d^3_{\sim} k_1 \ldots \int d^3_{\sim} k_j \, \bar{\rho}_j(k_1, \ldots k_j) \, \sigma_j(k_1, \ldots k_j), \tag{I-4}$$

with a 'bar' marking complex conjugation.

Field Operators

$\hat{\varphi}^{(+)}_{\alpha}(\underset{\sim}{k})$, and its adjoint $\hat{\varphi}^{(-)}_{\alpha}(\underset{\sim}{k})$ (α denotes the various particles, adjoints, and components of the Field and will be dropped for this outline (for example: electrons and positrons, spin up and spin down)), are operator valued distributions on $\mathcal{H}$, meaning that for a test function $\mu = \mu(\underset{\sim}{k})$ (which might be taken as infinitely differentiable with compact support, as is $\mu(x) = e^{-x^2}$ for $-L \leq x \leq L$),

$$\hat{\varphi}^{(\pm)}(\mu) = \int d^3_{\sim} k \; \mu(\underset{\sim}{k}) \; \hat{\varphi}^{(\pm)}(\underset{\sim}{k}) \tag{I-5}$$

are operators in $\mathcal{H}$ such that

$$\hat{\varphi}^{(+)}(\mu) \mid \sigma \rangle = \mid 0, \; \sigma_o \, \mu(k_1), \begin{Bmatrix} \text{Sym.} \\ \text{or} \\ \text{Anti-} \\ \text{Sym.} \end{Bmatrix} \sigma_1(k_1) \, \mu(k), \; \ldots \rangle \tag{I-6}$$

and

$$\hat{\varphi}^{(-)}(\mu) \mid \sigma \rangle = \mid \mu^* \sigma_1, \; \mu^* \sigma_2, \; \ldots \rangle \; , \tag{I-7}$$

where

$$\mu^* \sigma_j(k_1, \ldots, k_{j-1}) = \int d^3_{\sim} k_j \, \mu(k_j) \, \sigma_j(k_1, \ldots, k_j) \, . \tag{I-8}$$

Commutation relations such as

$$\langle 0 \,|\, \left[\hat{\varphi}^{(-)}(\bar{\sigma}), \; \hat{\varphi}^{(+)}(\rho)\right] \,|\, 0 \,\rangle = \int d^3_{\sim} k \; \overline{\sigma(k)} \; \rho(k) \tag{I-9}$$

are then derived.

N-Particle States and Propagators

A one-particle state $|\, \rho_1 \rangle$ is defined as

$$|\, 0, \; \rho_1(k_1), \; 0, \; \ldots \,\rangle \, , \tag{I-10}$$

and describes a universe of one particle with momentum probability density $|\, \rho_1(k_1)|^2$. $\rho_1(k_1)$ appears in Schroedinger's theory as a wave function. Similarly, an N-particle state $|\, \rho_N \rangle$ can be defined as N particles having momentum probability density $|\, \rho_N(k_1, \ldots, k_N)|^2$. Generalized state vectors can be written formally as

$$|\, \underset{\sim}{k} \,\rangle = \hat{\varphi}^{(+)}(\underset{\sim}{k}) \,|\, 0 \,\rangle = |\, 0, \; \delta(k_1 - k), \; 0 \ldots \,\rangle \, . \tag{I-11}$$

Lorentz invariant operators, defined by

$$\hat{\varphi}^{(+)}(k) = \sqrt{2k^0} \; \hat{\varphi}^{(+)}(\underset{\sim}{k}), \; k^0 = + \sqrt{m^2 + k^2}, \quad \text{and} \tag{I-12}$$

satisfying operator-valued distributions,

$$\hat{\varphi}^{(+)}(x) = \frac{1}{(2\pi)^{2/3}} \int \frac{d^3k}{2k^0} \, e^{ik\cdot x} \, \hat{\varphi}^{(+)}(k)$$

$$= \underset{n \to \infty}{\text{Limit}} \, \varphi^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k^0}} \, e^{ik\cdot x} \, \hat{\varphi}^{(+)}(\underset{\sim}{k}) \, \tau_n(\underset{\sim}{k}) \, , \tag{I-13}$$

where $\tau_n(\underset{\sim}{k})$ is a sequence of smooth functions (i.e., $e^{\frac{-ik\cdot n}{|n|}}$) approaching the constant 1, enable commutators to be evaluated:

$$i \, \{\varphi^{(-)}(x), \, \varphi^{(+)}(y)\} = D^-(x-y) = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k^0} \, e^{-ik\cdot(x-y)}, \tag{I-14}$$

The negative frequency commutator, $D^-(x)$, exists as a generalized function in the sense that, for f belonging to a space of test functions (say $f(z) = e^{-c|z|^2} P(z)$, $P(z)$ polynominal and $c > 0$):

$$(D^-, f) = \int d^4x \, D^-(x) \, f(x) = \underset{n \to \infty}{\text{Limit}} \int d^4x \, D_n^-(x) \, f(x) \, ,$$

where

$$D_n^-(x) = \frac{i}{(2\pi)^3} \int \frac{d^3\underset{\sim}{k}}{2k^0} \, e^{-ikx - \frac{k^0}{n}} \tag{I-15}$$

By expanding into the complex plane,

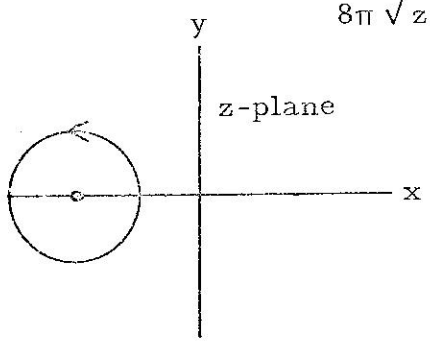
$$x = \begin{pmatrix} x^0 \\ \underset{\sim}{x} \end{pmatrix} \to z = \begin{pmatrix} z^0 \\ \underset{\sim}{z} \end{pmatrix} = \begin{pmatrix} x^0 \\ \underset{\sim}{x} \end{pmatrix} + i \begin{pmatrix} y^0 \\ \underset{\sim}{y} \end{pmatrix} = x + iy \, , \tag{I-16}$$

and by defining 4 dimensional surface contours ($s^{\pm}$) {enters causality}, strips of analyticity of f(z) define $\mathcal{D}(z)$ by

$$(D^-, f) = \int_{s^-} d^4z \; \mathcal{D}(z) \; f(z) \; ; \tag{I-17}$$

and $\mathcal{D}(z)$ can be evaluated (by taking limits) to be:

$$\mathcal{D}(z) = \frac{m}{8\pi\sqrt{z^2}} \; H_1^{(1)} \left(m\sqrt{z^2} \right), \tag{I-18}$$



$z^2 = g_{\mu\nu} z^{\nu} = x^2 - y^2 + 2i \cdot x \cdot y$, where $H_1^{(1)}$ is the first order Hankel function.

s^- (negative frequency contour)

The equation

$$D^-(x) = \frac{1}{(2\pi)^4} \int d^4p \; \frac{e^{ihx}}{m^2 - p^2} = \frac{1}{(2\pi)^4} \int d^4p \; \mathcal{D}(p) \; e^{ihx} \tag{I-19}$$

defines

$$\mathcal{D}(p) = \frac{1}{m^2 - p^2} = \frac{1}{m^2 + \underset{\sim}{p}^2 - (p^0)^2} \tag{I-20}$$

as a "particle propagator" (here as a scalar meson of mass m). Thus we see how vacuum state expectation values of commutators lead to 'propagators' which 'mediate' interactions between other particles.

Lagrangian

The 'Lagrange density'

$$\mathcal{L}(x) = \mathcal{L}(\psi_\alpha(x), \partial\psi_\alpha(x)) \tag{I-21}$$

[α again denotes various particles, adjoints and components, and ∂ refers to partial derivatives with respect to space and time] is an algebraic scalar combination of the various field operators, their adjoints, and first derivatives of these quantities.

The Action operator for a world region R is defined as

$$A(R) = \int_R d^4x \; \mathcal{L}(x) \; . \tag{I-22}$$

The condition (analogous to the classical Hamiton principle) that

$$\delta A(R) = 0 \tag{I-23}$$

for every region R, and every variation $\psi_\alpha \to \psi_\alpha + \delta\psi_\alpha$ which vanishes on the boundary of R, implies the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} \right) - \frac{\partial \mathcal{L}}{\partial \psi_\alpha} = 0 \; . \tag{I-24}$$

Symmetries of a continuous group G (usually the Lorentz group) are imposed upon the Lagrangian by insisting that the group of transformations g,

$$g \text{ such that } \quad x^\mu \to x^{\mu'}$$

$$\psi_\alpha(x) \to \psi_\alpha'(x') \; ,$$

leaves the Action in the same region R unchanged.

The observables, or dynamic variables of the system, corresponding to a group of continuous symmetry transformations with s parameters:

$$x^{\mu} \xrightarrow{g} x^{\mu'} = x^{\mu} + \sum_{j=1}^{s} \Lambda^{\mu}_{j} \, \delta\omega^{j}$$

$$\psi_{\alpha}(x) \xrightarrow{g} \psi_{\alpha}'(x') = \psi_{\alpha}(x) + \sum_{i} \Omega_{\alpha i} \, \delta\omega^{i} \, , \tag{I-25}$$

are calculated to be:

$$\mathcal{G}^{\mu}_{j} = \Lambda^{\mu}_{j} \, \mathcal{L} + \sum_{\alpha} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi_{\alpha})} (\Omega_{\alpha j} - \Lambda^{\nu}_{j} \, \partial_{\nu} \psi_{\alpha}) \, ,$$

$$\text{for} \qquad j = 1, \ldots, s \, . \tag{I-26}$$

For example, under the translation group, T:

$$x^{\mu} \xrightarrow{t} x^{\mu'} = x^{\mu} + \delta\omega^{\mu}$$

$$\psi_{\alpha}(x) \xrightarrow{t} \psi_{\alpha}'(x') = \psi_{\alpha}(x) \, , \tag{I-27}$$

and the energy-momentum stress tensor is obtained:

$$\mathcal{G}^{\mu\nu} = \mathcal{T}^{\mu\nu} = \delta^{\mu\nu}\mathcal{L} - \sum_{\alpha} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi_{\alpha})} \, \partial^{\nu}\psi_{\alpha} \, . \tag{I-28}$$

The Lagrangian density is finally defined by using the Normal Ordering symbol : ... : meaning "situate all annihilation operators

(φ^-) to the left of all creation operators (φ^+), " and by linearly decomposing $\varphi = \varphi^- + \varphi^+$ (negative and positive frequency components). For example:

$$:\varphi\psi\chi: = \varphi^+\psi^+\chi^+ + \varphi^+\chi^+\chi^- + \varphi^+\chi^+\psi^- + \psi^+\chi^+\varphi^-$$
$$+ \varphi^+\psi^-\chi^- + \psi^+\varphi^-\chi^- + \chi^+\varphi^-\psi^- + \varphi^-\psi^-\chi^- \quad . \qquad \text{(I-29)}$$

For the free uncharged meson field, the Lagrangian

$$\mathcal{L}(x) = \frac{1}{2} : [\partial^\mu \varphi(x)\, \partial_\mu \varphi(x) - m^2 \varphi^2(x)] : \qquad \text{(I-30)}$$

insures that the corresponding Euler-Lagrange equation is the Klein-Gordon equation for φ:

$$(\Box_x - m^2)\, \varphi(x) = 0 \quad . \qquad \text{(I-31)}$$

One can calculate the expectation value of the space-displacement operator:

$$\langle \rho_1 | k^\mu | \rho_1 \rangle = \int d^3\underset{\sim}{\ell} \int d^3\underset{\sim}{k}\, \bar{\rho}_1(\underset{\sim}{\ell})\, \rho_1(\underset{\sim}{k})\, k^\mu \langle 0 | \varphi^{(-)}(\underset{\sim}{\ell})\, \varphi^{(+)}(\underset{\sim}{k}) | 0 \rangle$$
$$= \int d^3\underset{\sim}{k}\, k^\mu\, |\rho_1(\underset{\sim}{k})|^2 \quad , \qquad \text{(I-32)}$$

suggesting the interpretation of $|\rho_1(\underset{\sim}{k})|^2$ as the probability density for a one-particle state to have momentum $\underset{\sim}{k}$.

Scattering Operator

The scattering operator $\int(k_1, \ldots, k_n, p_1, \ldots, p_n)$ enables one to construct in and out fields[2] which describe 'eigen-fields' of a

Lagrangian which includes interaction terms of two or more different fields. The fields considered in this work are those of massive, strongly interacting particles. Mass zero fields require special attention due to additional constraints (gauge groups, etc.).[4] Given a Lagrange density $\mathcal{L}(x) = \epsilon \Lambda(x)$, imposing Lorentz invariance, causality, and unitarity (specification of a complete set of scattering states) just about completely determines S via recursion relations[5] (on S_n, to follow) in a perturbation representation:

$$S = I + \sum_{n=1}^{\infty} \epsilon^n \int d^4x_1 \ldots \int d^4x_n \, S_n(x_1, \ldots, x_n) \qquad \text{(I-33)}$$

["just about" is inserted because of troubles related to renormalization difficulties at the point $(x_1, \ldots, x_n) = (0, \ldots, 0)$. This problem, however, has recently been solved[6] without additional parameters by defining a correct "principle value" for integrals]. It is therefore expected that any reasonable scattering theory which satisfies causality and Lorentz invariance (if a relativistic theory) and further which unitarizes $\mathcal{L}(x)$ properly would uniquely define S.

It should, therefore, come as no surprise that Dispersion Theory which closely "approximates" these conditions (unitarization of an interaction amplitude) would agree with experimental data if the correct Lagrangian is somehow inserted as input to describe the dynamics. Scotti and Wong[1] appear to have accomplished this. They therefore have included the correct Lagrangian interaction, and proper

insertion of this same physics into Schroedinger theory should give excellent results, especially as unitarization in the latter theory is absolutely correct.

The scattering operator is given by the well-known expression:

$$S(t) = \mathcal{T} \exp\left[i \int d^4 x_1 \, \mathcal{L}(x_1) \right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4 x_1 \ldots \int d^4 x_n \; \mathcal{T}\{\mathcal{L}(x_1), \ldots, \mathcal{L}(x_n)\}, \tag{I-34}$$

where $\mathcal{T}$ is the time-ordering operator acting on all x_j^0, $j = 1, \ldots, n$:

$$\mathcal{T}\{\varphi(x_1)\,\varphi(x_2)\} = \begin{cases} \varphi(x_1)\,\varphi(x_2) & t_1 \geq t_2 \\ \varphi(x_2)\,\varphi(x_1) & t_2 > t_1 \end{cases} . \tag{I-35}$$

[As written as $\int d^4 x_j = \int_0^\infty dx_j \int_0^\infty dy_j \int_0^\infty dz_j \int_{-\infty}^{t_j} dt'_j$, (I-36)

which is the form used by Dyson. Goldstone, however, defines

$$\int d^4 x_j = \int_0^\infty dx_j \int_0^\infty dy_j \int_0^\infty dz_j \int_{-\infty}^{t_j} dt'_j \; , \tag{I-37}$$

where $t_{j+1} > t_j$, $j = 1, \ldots, n$, and the $\frac{1}{n!}$ factor is omitted. This form explicitly displays the time ordering which is essential to his proof of the "linked-cluster" expansion.]

Wick's Theorem considerably simplifies the algebra in manipulating such expressions. It relies on the fact that the quantity defined as the contraction of A and B (the propagator),

$$A \cdot B \cdot = T[AB] - :AB: \tag{I-38}$$

is a "c-number", and that the time ordered product can be reduced to sums of products of contractions and normal products. Only one surviving term consisting of contractions can give a finite number to the expectation value of S in the vacuum state. Time ordering of fields operators give rise to commutators which physically represent interactions via particle exchange. [i.e., $\hat{D}(p)$] At this stage a loose link between the Lagrangian theory and the Hamiltonian theory can be made (without much mention of deeper theorems concerned with transforming from free to interacting fields). Classically:

the Lagrangian $\quad L = T$ (kinetic energy) $- V$ (interaction energy),

$$= L_0 + L_{INT}$$

and the Hamiltonian $\quad H = T + V = H_0 + H_{INT}.$ (I-39)

In the Lagrangian theory, the only contribution to that part of the scattering matrix causing transitions is the interaction part of the Lagrangian. In the Hamiltonian formalism, a Heisenberg representation is chosen for the state vectors, so that only the interaction part of the Hamiltonian affects the scattering (or equivalently, the development) operator:

$$S(t) = \mathcal{T} \exp\left(-i H_{INT}\, t\right) \tag{I-40}$$

It remains to compare the Lagrangian density with H_{INT}. For two particles scattering via some mutual interaction:

$$\int d^4x\, H(x)_{INT} = -\frac{1}{2} \int d^4x_1 \int d^4x_2 \;\; \mathcal{T}\{\mathcal{L}(x_1)\, \mathcal{L}(x_2)\}_{INT} \tag{I-41}$$

[i.e., $\mathcal{T}$ ( )--|--( ) ≡)----(, or $\mathcal{L}_{INT}\, \mathcal{L}_{INT} = H_{INT}$].

For the Hamiltonian formalism, the adiabatic theorem of Gell-Mann and Low[7] establishes a scattering matrix for an interaction which is "slowly" turned on and off. There does not exist any such "neat" theorem for the Lagrangian formalism.

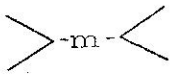
Appendix C contains an outline of the derivation of the many-body problem to be used in the nuclear matter problem.

Graphs

To facilitate calculations and physical understanding of these formalisms, various graphical procedures have been developed. The physics selects graphs contributing to a calculation by the proper choice of initial and final state vectors. For the Lagrangian formalism which will be used here to calculate the potentials, (H_{INT}), Feynman rules will be used. For the Hamiltonian formalism, to be extended to formulate many-body theory of nuclear matter, Brueckner rules will be used. With the Brueckner rules, the Brueckner-

Goldstone "linked-cluster" expansion can clearly be seen to include all those graphs which are not composed of 'disconnected' parts.[8]

Feynman rules:



$i\, g(4\pi)^{1/2}$ each vertex

ψ each particle (nucleon)

each 'virtual' particle (meson) has propagator $\dfrac{1}{q^2 + m^2}$ $\times$ spin and isospin factors (q is the 4-momentum transfer between the nucleons).

Brueckner rules:

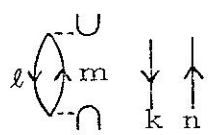
For n^{th} order graphs: draw n broken horizontal lines representing interactions and connect $2n$ end points with hole and particle lines, such that at every point, one line comes in, and one goes out.

Particle $\uparrow$; hole $\downarrow$; hole $\circlearrowleft$

Contributions to graph:

1. $\frac{1}{2} V_{mn,k\ell}$ for each vertex

2. each of $(n-1)$ intervals denotes a propagator, i.e.,



gives $(E_k + E_\ell - E_m - E_n)^{-1}$.

3. attach $(-)^{L+H}$ L = number of closed Fermi's lines

H = number of hole lines

4. number creation operator n_k for each hole,

$(1 - n_m)$ each particle; $[(n_k)^m = n_k$, by idempotency$]$.

II. POTENTIALS DERIVED FROM LAGRANGIAN

Using Feynman rules, the following Lagrangians will be considered:

π meson (pseudoscalar in coordinate-space and isovector in charge-space):

$$\mathcal{L}_\pi = (4\pi)^{1/2}\, g_\pi\, \bar{\psi}\, \gamma_5\, \tau \cdot \varphi_\pi\, \psi\,, \tag{II-1}$$

where g_π is the coupling strength of the meson field, φ_π, to the nucleon fields ψ and $\bar{\psi}$.

η meson (pseudoscalar in coordinate-space and isoscalar in charge-space):

$$\mathcal{L}_\eta = (4\pi)^{1/2}\, g_\eta\, \bar{\psi}\, \gamma_5\, \varphi_\eta\, \psi\,, \tag{II-2}$$

σ meson (scalar in both coordinate and charge-spaces.) (The σ is not yet established to be a bona fide meson,[9] in which case it may represent a strong π-π, S-wave resonance[1]):

$$\mathcal{L}_\sigma = (4\pi)^{1/2}\, g_\sigma\, \bar{\psi}\, \varphi_\sigma\, \psi\,, \tag{II-3}$$

ρ meson (vector in both coordinate and charge-spaces):

$$\mathcal{L}_\rho = i\,(4\pi)^{1/2}\,(g_{\rho 1} + g_{\rho 2})\, \bar{\psi}\, \gamma_\nu\, \tau \cdot \varphi_\rho^\nu\, \psi$$

$$- (4\pi)^{1/2}\,(g_{\rho 2}/2m_\rho)\,(p + p')_\nu\, \bar{\psi}\, \tau \cdot \varphi_\rho^\nu\, \psi\,. \tag{II-4}$$

The tensor coupling constant, (or anomalous magnetic-current term), $g_{\rho 2}$, is also included as suggested by the experimentally determined magnetic moment form factors of the nucleon.[1]

ω and φ mesons (vectors in coordinate-space and scalars in charge-space):

$$\mathcal{L}_{\omega(\varphi)} = i(4\pi)^{1/2} \, g_{\omega(\varphi)} \, \bar{\psi} \, \gamma_\nu \, \varphi^\nu_{\omega(\varphi)} \, \psi \, . \tag{II-5}$$

[These five mesons (including the tensor coupling of the ρ also exhaust the five possible independent Dirac γ-matrices that describe the various meson exchanges.]

A spinor representation for ψ is then picked (consistent with the matrix representations for γ_5 and γ_ν):

$$\psi_{(\alpha)} = e^{i p \cdot x} \, u_\lambda^{(1,2)}(p) \tag{II-6}$$

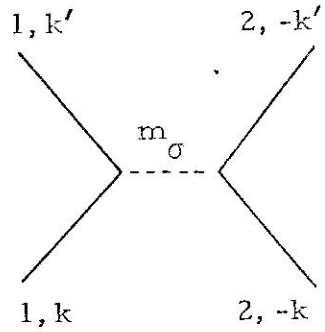
where λ designates + or - helicity, and 1 or 2 designates positive or negative energy.

i.e.

$$u_+^1(p) = \sqrt{\frac{E+m}{2E}} \begin{bmatrix} 1 \\ 0 \\ \frac{\sigma \cdot p}{E+m} \\ 0 \end{bmatrix} , \quad u_+^2 = \sqrt{\frac{E+m}{2E}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{-\sigma \cdot p}{E+m} \end{bmatrix} ; \tag{II-7}$$

$E = \sqrt{p^2 + m^2}$, and m designates the rest mass of the nucleon, taken to be the same for protons and nucleons.

For example, according to the rules of Section 2, one obtains a "potential" [before a non-relativistic reduction], for the scalar meson (σ) to be:



(plus exchange)

$$M(k,k) = -\frac{(E'+m)(E+m)}{4m^2}\left[1 - \frac{\sigma_1 \cdot k' \sigma_1 \cdot k}{(E'+m)(E+m)}\right] \times \frac{g_\sigma^2}{\Delta^2 + m_\sigma^2}\left[1 - \frac{\sigma_2 \cdot k' \sigma_2 \cdot k}{(E'+m)(E+m)}\right], \qquad \text{(II-8)}$$

where $\Delta^2 = (k-k')^2$ is the square of the momentum transferred via the σ meson. ($p = \hbar k$, but as $\hbar$ will be set $\equiv 1$, the two will be used interchangeably.) (For inelastic scattering, the 'propagator' is:

$$\frac{1}{\Delta^2 - (E_f - E_i)^2 + \mu^2} \quad .$$

For nuclear matter, where inelastic scattering can take place, one can only hope that $|E_f - E_i| << \mu^2$. Also, see the comment about $\frac{\sigma \cdot p \times p'}{m^2}$ made at the end of the section on Nuclear Matter).

There are five independent such amplitudes for each meson, depending on the final and initial z-components of the total spin of the nucleons.

Partial-wave Amplitude	Corresponding Helicity Amplitude		
		In	Out
singlet $J = L$	1.	$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$
triplet $J = L$	2.	$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$
coupled $J = L \pm 1$	3.	$\downarrow\downarrow$	$\downarrow\downarrow$
(triplet)	4.	$\uparrow\uparrow$	$\uparrow\uparrow$
	5.	$\downarrow\downarrow$	$\uparrow\uparrow$
	(6.	$\uparrow\uparrow$	$\downarrow\downarrow$)

(equal to the amplitude for elastic scattering 5. due to invariance under time-reversal)

For elastic scattering, there are also five independent "Wigner-amplitudes"[10] that can be constructed out of bilinear combinations of the spin and momenta (to order p^2). It is 'wise' to pick:

$$1;\ \sigma_1 \cdot \sigma_2;\ (r \times p) \cdot S;\ (3\sigma_1 \cdot \hat{r}\ \sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2)(= S_{12});$$

$$\sigma_1 \cdot p\ \sigma_2 \cdot p / m_n^2 \qquad \text{(II-9)}$$

The spin (or helicity) amplitudes can easily be projected from the Wigner set by the use of bessel transforms with respect to momentum-transfer (Δ):[11]

$$M_{(1;\sigma_1\cdot\sigma_2;\sigma_1\cdot p\sigma_2\cdot p)} = -\frac{m}{4\pi}\int \underset{\sim}{dr}\; u(r,\Delta)_{(1;\sigma_1\cdot\sigma_2;\sigma_1\cdot p\sigma_2\cdot p)} \exp(-i\Delta\cdot r)$$

$$= -\frac{m}{4\pi}\int u_{(\;)}\; 4\pi \sum_{\ell,k} (-i)^{\ell}\, j_{\ell}(\Delta r)\, \times$$

$$Y_{\ell k}^{*}(\Omega_r)\, Y_{\ell k}(\Omega_\Delta)\; r^2\, dr\, d\Omega_r\,, \qquad \text{(II-10)}$$

and because $\int d\Omega_r = \delta_{\ell 0}$:

$$M_{(1;\sigma_1\cdot\sigma_2;\,\sigma_1\cdot p\sigma_2\cdot p)} = -m\int u_{(\;)}\, j_0(\Delta r)\, r^2\, dr \qquad \text{(II-11)}$$

[$M_{(\sigma_1\cdot L\sigma_2\cdot L)}$ is projected from a linear combination of $\int u_{(\sigma_1\cdot L\sigma_2\cdot L)}\, j_0$ and $\int u_{(\sigma_1\cdot L\sigma_2\cdot L)}\, j_2$, and is therefore not uniquely transformable.]

$$M_{S_{12}} = -\frac{m}{4\pi}\int u_{S_{12}}(r) S_{12}(r)\, \exp(-i\Delta\cdot r)\, \underset{\sim}{dr}$$

$$= -\frac{m}{4\pi}\int dr\, r^2\, u_{S_{12}} \int \left(\frac{24\pi}{5}\right)^{1/2} \sum_{n=-2}^{n=+2} (-)^n\, Y_{2n}(\Omega_r)$$

$$\times\, [\sigma_1\times\sigma_2]^{(2)}_{-n}\, \exp(-i\Delta\cdot r)\, d\Omega_r$$

$$= -\frac{m}{4\pi}\int dr\, r^2\, u_{S_{12}}\, (-4\pi)\, j_2(\Delta r)\, \left(\frac{24\pi}{5}\right)^{1/2} \times$$

(Equation (II-12) continued on next page.)

$$\times \sum_{n=-2}^{+2} (-)^n \, Y_{2n}(\Omega_\Delta) \, [\sigma_1 \times \sigma_2]^{(2)}_{-n}$$

$$= (-S_{12}(\Delta)) \, (-m) \int u_{S_{12}}(r) \, j_2(\Delta r) \, r^2 \, dr \qquad \text{(II-12)}$$

$$M_{LS} = \frac{m}{4\pi} \int \exp(-ik_f \cdot r) \, u^{(r)}_{LS}(S \cdot r \times (-i\Delta_{\underset{\sim}{r}}) \exp(ik_i \cdot r)) \, d\underset{\sim}{r}$$

$$= \frac{-m}{4\pi} \int dr \, r^2 \, u_{LS}(r) \int (S \cdot r \times k_i) \exp(-i\Delta \cdot r) \, d\Omega_r$$

$$= \frac{-m}{4\pi} \int dr \, r^2 \, u_{LS}(r) \, i \, 4\pi r j_1(\Delta r) \, (S \cdot k_i \times q) / |q|$$

$$= i \frac{1}{q} (S \cdot k_i \times k_f) \, (-m) \int u_{LS}(r) \, j_1(\Delta r) \, r^3 \, dr$$

$$= S \cdot n \, ik^2 \, \frac{\sin\theta}{2q} \, (-m) \int u_{LS}(r) \, j_1(\Delta r) \, r^3 \, dr \, , \qquad \text{(II-13)}$$

where $\quad n = \underset{\sim}{k_i} \times \underset{\sim}{k_f} \, / \, |k_i \times k_f| \, , \; \underset{\sim}{q} = \underset{\sim}{k_f} - \underset{\sim}{k_i} \, ,$

and $\quad |k_i \times k_f| = k^2 \sin\theta$

Thus, assuming:

$$M = M_1 + M_{\sigma_1\sigma_2} \, \sigma_1 \, \sigma_2 + M_{S_{12}} \, S_{12} + M_{LS} \, L \cdot S + M_{\sigma \cdot p} \, \sigma_1 \overleftrightarrow{\cdot \, p\sigma_2 \cdot} \, p, \qquad \text{(II-14)}$$

one can calculate:

$$f_\lambda(\Delta, p^2) = \sum_{\substack{i=1, \sigma_1\sigma_2, S_{12}, \\ LS, \sigma\cdot p}} a_i^\lambda(\Delta)\, M_i(\Delta, p^2) \qquad \text{(II-15)}$$

where f_λ is one of the five invariant helicity amplitudes which have been found from the amplitudes for meson exchange.[11] These five equations are then inverted to find

$$M_i(\Delta, k^2) = \Sigma [a^{-1}]_\lambda^i \; f_\lambda(\Delta, k^2) \qquad \text{(II-16)}$$

The arrows over the $\sigma \cdot p$ term indicate the directions of operation. As written $\sigma \cdot p$ is averaged after operating to both the left and right.

Finally, one takes inverse Bessel transforms to find $M_i(r, k^2)$. The five Wigner operators are eigenoperators of eigenfunctions of the <u>total</u> angular momentum $(L + S = J)$, and by expressing the potential by means of this set, the M_i lend themselves, trivially, to a partial-wave reduction. For this purpose $\sigma_1 \cdot p\sigma_2 \cdot p$ is given in a more useful form in Appendix B.

The non-relativistic reduction, to get a potential $V(r, p^2)$ (any higher powers of p^2 used in Schroedinger's equation would obviously require additional boundary conditions than assumed for the free particles) is done by expanding all powers of

$$\frac{E}{M} = \frac{(k^2 + m^2)^{1/2}}{m} \simeq 1 + \frac{k^2}{2m^2} + \frac{k^4}{6m^4} + \frac{k^6}{24m^6} + \ldots , \qquad \text{(II-17)}$$

and by keeping factors like $\cos\theta = 1 - \frac{\Delta^2}{2k^2}$ and $\frac{1}{k^2}$ which occur in the b_λ. A phase space factor of $\frac{m}{(m^2 + k^2)^{1/2}}$ is also expanded to $1 - \frac{k^2}{2m^2}$, allowing local potentials to be defined.[11] Then all powers higher than $\frac{k^2}{m^2}$ are dropped. This reduction is thus valid for $\frac{k^2}{m^2} << 1$. The velocity dependence is seen to arise from factors of $\frac{\sqrt{k^2 + m^2}}{m}$ and products of $\sigma \cdot k$ [combinations of which are determined by the type of meson being exchanged] from the nucleon spinors.

To remove the δ-function singularities at $r = 0$ due to infinities in momentum transfer space $\left[\int \frac{\Delta^n j_\ell(\Delta r)}{\Delta^2 + \Lambda^2} d\Delta,\right.$ diverges for $\left. n \geq 0 \right]$, a regularization in the form of products of Feynman-type propagators $\left(\prod_i \frac{\Lambda_i^2 - \mu^2}{\Lambda_i^2 + \Delta^2}\right)$ multiplying the integrands of integrals over momentum-transfer space was introduced. $\prod_i (\Lambda_i^2 - \mu^2)$ was chosen as a factor to enable g^2 for each meson to have the usual meaning $\left[V \sim \frac{g^2 e^{-\mu r}}{r}\right]$ in the asymptotic region. Four Λ_i's were needed to take care of the highest power of Δ^6 found in the central part of the "induced-tensor" term of the ρ meson. For example, to regularize:

$$\int \frac{\Delta^2 \, j_\ell(\Delta r)}{\Delta^2 + \mu^2} \, d\Delta ,$$

introduce $\dfrac{\Lambda^2}{\Delta^2 + \Lambda^2}$:

$$\int \frac{d\Delta \; \Delta^2}{\Delta^2 + \Lambda^2} \; \frac{\Lambda^2}{\Delta^2 + \Lambda^2} \, j_\ell(\Delta r) = \int \frac{[(\Delta^2 + \mu^2) - \mu^2] \; \Lambda^2 \, j(\Delta r) \, d\Delta}{(\Delta^2 + \mu^2)(\Delta^2 + \Lambda^2)}$$

$$= \Lambda^2 \int \frac{d\Delta \; j_\ell(\Delta r)}{\Delta^2 + \Lambda^2} - \mu^2 \Lambda^2 \int d\Delta \, j_\ell(\Delta r) \left[\frac{1}{\Lambda^2 - \mu^2} \; \frac{1}{\Delta^2 + \mu^2} \right.$$

$$\left. + \frac{1}{\mu^2 - \Lambda^2} \; \frac{1}{\Delta^2 + \Lambda^2} \right] . \qquad \text{(II-18)}$$

A look at the various Lagrangians and at the nucleon spinors show that, because the spinors are well-behaved functions of k_i and k_f, only the ρ meson derivative coupling term should have any divergences. However, when expanding the spinor normalization factors,

$$\frac{\Delta^2}{k^2 + m^2} \sim \frac{\Delta^2}{m^2} \left[1 - \frac{k^2}{m^2} \right] \qquad \text{(II-19)}$$

such divergences "appear". The 'masses' used to regularize these divergences were, however, larger than the nucleon mass which imposed the harshest cutoff restrictions as explained below.

To insure having an analytic potential at small distances of r,

and to remedy the $1/r^3$ behavior of the $L \cdot S$ and S_{12} terms, all potentials in coordinate space were multiplied by $(1 - e^{-r/m_n})^3$. (After regularization, higher order products of propagators do not necessarily change the 'order' of the potential at $r = 0$, but just introduce more curvature in the potential at small coordinate space distances.) As these velocity-dependent potentials are expansions in $\frac{k^2}{m_n^2}$, and as the highest mass of mesons considered is on the order of 1 Bev [φ meson], they should not be considered as having any meaning within a distance of $\sim 1/m_n$, or .2 fermi. This explains the choice of cutoff to remove the divergences in coordinate space. Also, the potential within $r < \frac{1}{m_n}$ was set equal to its value at $r = \frac{1}{m_n}$ to aid the numerical solution of the phase shifts. For the same physical reasons, the cutoffs used in the regularizations should be on the order of 1 Bev (but greater than the mass of the φ meson). This also influenced the choice of meshes which was determined by insuring that, within each interval at least several points spanned the Compton wave length of the most predominant meson in that interval.

Mention should be made that the cutoffs recommended in Ref. 11

$$\left(\int_0^\infty d\Delta \simeq \int_{m_n}^\infty d\Delta \right)$$

give rise to unphysical oscillatory potentials dying off as inverse powers of r, for large values of r.

The natural way to symmetrize the velocity-dependent k^2 terms is:

$$\widetilde{k^2 V} = \frac{1}{2}(k^2 V + V k^2) . \qquad \text{(II-20)}$$

However, the $\sigma_1 \cdot k\, \sigma_2 \cdot k$ terms could be written as:

$$\frac{1}{2}(\sigma_1 \cdot k\, \sigma_2 \cdot k\, V + V \sigma_1 \cdot k\, \sigma_2 \cdot k) \qquad \text{(II-21)}$$

(again symmetrical in spins 1 and 2) or as:

$$\frac{1}{2}(\sigma_1 \cdot k\, V\, \sigma_2 \cdot k + \sigma_2 \cdot k\, V\, \sigma_1 \cdot k) \qquad \text{(II-22)}$$

admitting some arbitrariness in the symmetrization. The first form was chosen as it coincides with the natural way the momentum-square, k^2, is treated. Anyway, this was done before fitting the experimental data, and the resultant effects would seem to be unimportant.

The $1/r^3$ singularities are quite unphysical. The attractive tensor potential used without a cutoff would predict no lowest eigenvalue for the two-nucleon system which, of course, can exist as a deuteron. Such a singularity, while clearly of dynamical origin in the case of the $L \cdot S$ force, and therefore as suspect as the rest of the potential in this small r region, is, however, purely static and physically necessary in the case of the tensor force. This cutoff in a purely

mechanical problem was first noticed by Hans Bethe[12] in 1940.

The final potentials may be found in Appendix A.

There does exist the question of other non-relativistic corrections, primarily the p^4 correction to the kinetic energy

Schroedinger's equation:

$$\left[\frac{p^2}{m} + V(r)\right] \psi(r) = E\,\psi(r) \quad , \tag{II-23}$$

for $E \sim 0$ suggests that $\frac{p^4}{m^2} \sim \frac{p^2}{m} V$, or that the kinetic energy correction is the same order of magnitude as the velocity-dependent potential. But this appears to be true only on the average. As will be seen, for large r, V(r) in Eq. (II-23) is small, and $\frac{p^2}{m}$ is therefore small, and therefore both $\frac{p^4}{m^2}$ and $\frac{p^2}{m} V$ are small and negligible. For small r, V and $\frac{p^2}{m} V$ are large, but ψ is ~ 0, and for low energies the effects of this small r region are unimportant. For larger energies, within the energy range being considered, this last statement for small r is probably also true. Anyway, this region is acknowledged to be phenomologically treated. For large r, regardless of the energy, it is shown that the velocity-dependent function is orders of magnitude smaller than the static term, so that there is no inconsistency in neglecting the kinetic energy correction.

III. SCATTERING MATRIX

Schroedinger's equation,

$$\left(\frac{-\hbar^2}{m}\nabla^2 + F(r)\right)\psi_E^+(r) = E\,\psi_E^+(r)\,, \tag{III-1}$$

can be recast into an integral equation:

$$\psi^+(r) = \varphi + \int G^+(\,r - r'\,)\,\frac{2m}{\hbar^2}\,F(r')\,\psi^+(r')\,dr'\,, \tag{III-2}$$

where $\varphi(r) = e^{ik\cdot r}$ is a plane wave and $G^+ = -\dfrac{1}{4\pi\,|r - r'|}\,e^{ik\cdot(r - r')}$ is the Green's function for the problem (assuming outgoing spherical waves as a boundary condition).

The procedure to solve:

$$(\nabla^2 + k^2)\,\psi_k^+(kr) = F(r)\,\psi_k^+(kr) \tag{III-3}$$

is to first solve

$$(\nabla^2 + k^2)\;\varphi_k(r) = 0\;, \tag{III-4}$$

then to solve $(\nabla_r^2 + k^2)\,G_k^+(r, r') = \delta(r - r')$, where G_k^+ is determined by a sum over the eigenfunctions determined by the solution of the

homogeneous equation. This procedure leads to the wave equation:

$$\psi^{+}(kr) = \varphi_k(r) + \int dr' \, G_k^{+}(r, r') \, F(r') \, \psi^{+}(kr') \, , \tag{III-5}$$

which expresses ψ^{+} as a sum of the homogeneous and inhomogeneous solutions of the differential equation. The normalization corresponding to this integral equation is

$$\int \varphi(r)^{*} \, \psi(r) \, dr = \int \varphi(r)^{*} \, \varphi(r) \, dr = 1 \, .$$

For $F(r)$ rotationally invariant, a partial wave reduction may be made to find δ_ℓ:

$$\psi_\ell^{+} = j_\ell + \int G_\ell^{+} \, F_\ell \, \psi_\ell^{+} \, ,$$

$$\left(\psi^{+}(kr) = \sum_\ell (2\ell + 1) \, P_\ell(\hat{k} \cdot \hat{r}) \, \psi_\ell^{+}(kr) \right) , \tag{III-6}$$

giving $\delta_\ell(k)$, the phase shift, determined by:

$$e^{i\delta_\ell} \sin\delta_\ell = -\frac{m}{k} \int j_\ell \, F_\ell \, \psi_\ell^{+} \tag{III-7}$$

Similarly:

$$\psi_\ell = j_\ell + \int G_\ell \, F_\ell \, \psi_\ell \, , \left[G_\ell(k, r', r) = \sum_{k'} \frac{j_\ell(k'r') \, j_\ell(k'r)}{k^2 - k'^2} = j_\ell \eta_\ell \right] \tag{III-8}$$

gives $\tan \delta_\ell = -\frac{m}{k} \int j_\ell \, F_\ell \, \psi_\ell$, where G_ℓ is the Green's function with standing wave boundary conditions. Neglecting spin, for readability, the standing wave solution is:

$$\psi_\ell(kr) \xrightarrow[r \to \infty]{\lim} j_\ell(kr) - \eta_\ell(kr) \tan \delta_\ell(k) \ . \qquad \text{(III-9)}$$

(j_ℓ and η_ℓ are Bessel and Neumann functions,[13] respectively.)

However, the potential derived in Section II is velocity-dependent:

$$(\nabla^2 + k^2)\ \psi_k(r) = V(r, \nabla)\ \psi_k(r)\ , \qquad \text{(III-10)}$$

and to be useful for nuclear matter calculations (see Appendix C), a correct integral-equation representation must be found.

If one 'retransforms' the differential equation to resemble Schroedinger's equation with a pure radial function as a potential, the potential becomes a function of k,[14] thereby destroying the eigenvalue properties of the integral equation. Alternative forms leave the homogeneous equation and the Green's function equation with a complicated function multiplying the Laplacian operator which destroys the usefulness of the simple Green's function solutions, and physical intuition gained when using a static potential. For example, the differential equation

$$\left[\nabla^2 + k^2 - \left(X(r) + Y(r)\frac{d}{dr} + Z(r)\frac{d^2}{dr^2}\right)\right]\psi = 0 \qquad \text{(III-11)}$$

can certainly be solved by numerical means (for X, Y, Z well-behaved). However, previous authors have, by dropping the $\sigma \cdot p$ terms (allowing $Y = 0$), recast this equation into the form:

$$\left[\left(1 + Z\right) \nabla^2 + k^2 - X - \frac{Z''}{2}\right] \psi = 0 \; ; \qquad \text{(III-11a)}$$

and some have rewritten

$$\left(\nabla^2 + \frac{k^2 - X - \frac{Z''}{2}}{1 + Z}\right) \psi = 0 \qquad \text{(III-12)}$$

as

$$\left(\nabla^2 - \frac{k^2 Z - X - \frac{Z''}{2}}{1 + Z} + k^2\right) \psi = 0 \; , \qquad \text{(III-13)}$$

giving an "energy-dependent" potential.

The correct procedure to 'solve'

$$(\nabla^2 + k^2)\, \psi(kr) = H(r, \nabla)\, \psi(kr) \qquad \text{(III-14)}$$

is to first solve:

$$(\nabla^2 + k^2)\, \chi(kr) = 0 \; , \qquad \text{(III-15)}$$

then solve:

$$(\nabla_r + k^2)\, G_k(r, r') = \delta(r - r') \; . \qquad \text{(III-16)}$$

One thereby obtains:

$$\psi(kr) = \chi(kr) + \int dr'\, G_k(r, r')\, \{H(r', \nabla_{r'})\, \psi(r, r')\} \; , \qquad \text{(III-17)}$$

or, after a partial wave reduction:

$$\psi_\ell = j_\ell + \int dr'\, j_\ell(r_<)\, \eta_\ell(r_>)\, \{H_\ell(r', \nabla_{r'})\, \psi_\ell(kr')\} \; , \qquad \text{(III-17a)}$$

($r_<$ is the lesser of r and r', and $r_>$ the greater),

which gives a formula for the phase shift:

$$\tan \delta_\ell(k) = -\frac{m}{k} \int dr \; j_\ell(kr) \; \{H_\ell(r, \nabla) \; \psi_\ell(kr)\} \quad , \qquad \text{(III-18)}$$

$$. \quad H(r, \nabla) = X(r) + Y(r) \frac{d}{dr} + Z(r) \frac{d^2}{dr^2}$$

for the potential forms derived in Section II. Similar coupled equations hold for the triplet $J = L \pm 1$ partial waves.

Integration by parts, for a potential well-behaved at the origin (say as a constant), and decreasing exponentially at infinity, yields:

$$(1 + Z_\ell) \; \psi_\ell = j_\ell + \int dr \; \psi_\ell \quad \left\{ X_\ell + Y_\ell \frac{d}{dr} + Z_\ell \frac{d^2}{dr^2} \right\} G_\ell \; , \qquad \text{(III-19)}$$

and

$$\tan \delta_\ell = -\frac{m}{k} \int dr \; \psi_\ell \left\{ X_\ell + Y_\ell \frac{d}{dr} + Z_\ell \frac{d^2}{dr^2} \right\} j_\ell \; , \qquad \text{(III-20)}$$

$$= -\frac{m}{k} \int dr \; \psi_\ell \; \overrightarrow{H_\ell}(\nabla, r) \; j_\ell \; .$$

Y_ℓ contains a $\frac{d}{dr}$ term from the $\sigma_1 \cdot p \sigma_2 \cdot p$ amplitude which previously could not be properly used in calculations using the "transformed equations" which is the probable reason many previous calculations just used four of the five possible independent potential forms. As the $G_\ell(r, r')$ and $j_\ell(r)$, and their derivatives, are simple functions of sines and cosines, these equations are no harder to solve than those using a potential depending solely on a radial parameter.

This was checked numerically by solving the $^1S_0$-state equation, first by a Runge-Kutta method for the differential equation, and then by using a combination of matrix inversion and perturbation methods for the integral equation. Only the potential used as defined here gave the correct wave function and phase shifts when comparison between the two solutions were made (i.e., formulations as that used by A. M. Green[14] were found to disagree). This served as both a check for the accuracy of the integral equation (meshes, etc.), and also as a verification of the importance of solving the correct eigenvalue problem presented above. In light of this, previous calculations involving energy-dependent potentials and/or energy-dependent boundary conditions should be reexamined as to their regions of validity.

The integrals were done using three regions of 10 point Gaussian quadratures:

1. $r = 0. - 3.$ $[\text{Bev}]^{-1}$
2. $r = 3. - 12.$ $[\text{Bev}]^{-1}$
3. $r = 12. - 30.$ $[\text{Bev}]^{-1}$

These 30 points were found to give the same accuracy as 47 points using Simpson's rule.

The conversion factor from fermis to $[\text{Bev}]^{-1}$ is $1\text{f} = 5.0686\ [\text{Bev}]^{-1}$ (1 fermi = 1 f = 1 fm = 1 fermometer = 10^{-15} m).

$$\left(\lambda\!\!\!^{-}_{n} = \frac{\hbar}{m_n c} [f] = \frac{\hbar c}{m_n c^2} = \frac{1}{m_n c^2} [\text{Bev}]^{-1} \right) .$$

The mass of the nucleon, m_n, is taken to be the averaged mass of the proton and neutron:

$$m_n = \frac{1}{2} [m_{proton} + m_{neutron}]$$

$$= \frac{1}{2} [2 m_{proton} + 1.3 \text{ Mev}]$$

$$= \frac{1}{2} [2(.93821) + 1.3 \times 10^{-3}] \text{Bev}$$

$$= .93886 \text{ Bev.}$$

To solve:

$$\psi = \chi + \int K\psi ,$$

matrix inversion was performed:

$$\psi = [1 - K]^{-1} \chi .$$

Because the diagonal elements of $[1 - K]$ were never zero, the pivot element in the matrix inversion procedure could be set equal to it, greatly reducing the time for matrix inversion to slightly over 2 seconds for a 30 by 30 matrix.

IV. NUCLEAR MATTER

The two nuclear properties sought in a nuclear matter calculation are the volume term of the Bethe-Weizaacker semi-empirical mass formula, and the uniform saturation density of large nuclei.[8]

The experimental binding energy can be taken to be $(-15.5 \pm 2.)$ Mev. The mean spacing between nuclei in a saturated large nucleus is taken to be $(1.07 \pm .03)$ fermi.

The calculations of nuclear matter, as performed by B-M,[15] were done for various potentials. It therefore seemed more fruitful to do a similar many-body calculation in order to directly compare the effects of functional forms of those potentials to this work. It was also hoped that the well-behaved potentials used would make the approximations used in B-M[15] more palatable.

The theory developed by K. A. Brueckner[15] describes the interactions of nucleons below the Fermi sea (lowest energy level of a system of fermions at zero temperature) via interactions of intermediate states of particles above the Fermi sea. The interactions of these particles above the Fermi sea correspond to repeated interactions of each pair ("ladder sum"), and also takes into account the forward scattering ("self-energy") of the excited "particles" with the unexcited "holes" of the nuclear medium, to infinite order (see Appendix C).

The equations that are solved are:

Green's functions:

$$G_\ell(r, r') = \frac{2}{\pi} \int_0^\infty \frac{k''^2 \, dk'' \, j_\ell(k''r) \, j_\ell(k''r') \, f(P, k'')}{2(E(k) - E^*(k''))} \qquad \text{(IV-1)}$$

for on-energy shell propagation, and with $\{2[E(k) - E^*(k'') - \Delta]\}$ as a denominator for off-energy shell propagation, with $E(k) = E(k_f)$ for $k \geq k_f$ due to averaging of the center of mass momenta.

Δ is an approximate mean excitation energy $= E(k_f) - E(0)$.

P is an average total momentum:

$$P^2 \equiv P^2_{AV} = \frac{12}{5} p_f^2 \left(1 - \frac{k}{p_f}\right)\left(1 + \frac{k}{2p_f} + \frac{k^2}{6p_f^2}\right) \Big/ \left(1 + \frac{k}{p_f}\right)$$

$$\text{for } k < k_f, \qquad \text{(IV-2)}$$

and $P^2 \equiv P^2_{AV} = 0$, for $k \geq p_f$.

$f(p, k'')$ is the angular average of the Pauli exclusion-principle projection operator:

$$f(p, k'') = 0 \qquad \left(k'' + \frac{p^2}{4}\right)^{1/2} < p_f \qquad \text{(IV-3)}$$

$$= 1 \qquad (k'' - p/2) > p_f$$

$$= \frac{k''^2 + \frac{p^2}{4} - p_f^2}{k'' p} \qquad \text{otherwise.}$$

The integral from zero to infinity was calculated as:[16]

$$\frac{2}{\pi}\int_0^{2.6\,p_f}\left[dk''k''^2\, j_\ell(k''r)\, j_\ell(k''r')\left\{\frac{f(p,k'')}{2[E(k)-E^*(k'')]}\right.\right.$$

$$\left.\left.-\frac{1}{2\left[E(k)-\frac{k''^2}{2m}\right]}\right\}\right]+\frac{2}{\pi}\int_0^\infty \frac{dk''k''^2\, j_\ell(k''r)\, j_\ell(k''r')}{2\left[E(k)-\frac{k''^2}{2m}\right]}\,, \qquad \text{(IV-4)}$$

because $E^*(k'')$ was set equal to $k''/2m$ for $k'' > 2.6\,p_f$. The last integral can be analytically evaluated:

$$\frac{2}{\pi}\int_0^\infty \frac{k^2\, j_\ell(kr)\, j_\ell(kr')}{z-\frac{k''^2}{m}} = ma\, h_\ell(iar_>)\, j_\ell(iar_<)\,,$$

$$\text{where } a = [-mz]^{1/2} \text{ (z is always } < 0); \qquad \text{(IV-5)}$$

j_ℓ and h_ℓ are ℓ^{th} order Bessel and Hankel functions, respectively with $r_>$ designating the greater value of r and r', and $r_<$ designating the lesser.

Wave-Functions:

$$u^{JS}_{\ell'\ell}(kr) = j_\ell(kr)\,\delta_{\ell'\ell} + \int_0^\infty dr'\, r'^2\, G_{\ell'}(r,r')$$

$$\times \sum_{\ell''=J-1}^{J+1} \overleftarrow{V}^{JS}_{\ell'\ell''}(r',\nabla)\, u^{JS}_{\ell''\ell}(kr')\,. \qquad \text{(IV-6)}$$

K Matrices:

$$K_{kk} = 4\pi \sum_{JS} \sum_{\ell=J-1}^{J+1} C_{J\ell S} \int_0^\infty r^2\, dr\, j_\ell(kr) \sum_{\ell'=J-1}^{J+1} \overleftarrow{V}^{JS}_{\ell\ell'}(r, \nabla)\, u^{JS}_{\ell\ell'}(r) \tag{IV-7}$$

$C_{J\ell S}$, the appropriate statistical weights, is given by:

$$\frac{(2T+1)\,(2J+1)}{3} \quad (=1 \text{ for S-states}),$$

T being the iso-spin, and J being the total angular momentum of the state in question.

Single Particle Potential:

$$\mathcal{V}(p) = \frac{6}{\pi^2} \int_0^{(p_f-p)/2} k'^2\, dk'\, (k' \,|\, K \,|\, k')$$

$$+ \frac{3}{\pi^2} \int_{|p_f-p|/2}^{(p_f+p)/2} k'^2\, dk' (k' \,|\, K \,|\, k') \left(1 + \frac{(p_f^2 - p^2 - 4k'^2)}{4pk'}\right) \tag{IV-8}$$

for $p < p_f$. For $p \geq p_f$, the first integral vanishes.

The self-consistent energy is

$$E(k) = \frac{k^2}{2m} + \mathcal{V}(k), \text{ and the binding energy per particle}$$

$$= \int_0^{k_f} \underset{\sim}{dk} \left(\frac{k^2}{2m} + \frac{1}{2}\mathcal{V}(k)\right) \Big/ \int_0^{k_f} \underset{\sim}{dk} \tag{IV-9}$$

The arrows on top of $\overleftarrow{V_{\ell'\ell''}}$ and $\overleftarrow{V_{\ell\ell'}}$ in the wave function and K-matrix equations means ∇ in $V(r, \nabla)$ operates on $G_{\ell'}$ and j_{ℓ}, respectively, according to the index of ∇_r. This is analogous to the treatment of the potential use in Section III for the proper treatment of velocity-dependent potentials.

At this point mention should be made that when calculating off the energy shell (inelastic scattering), a new, sixth, possible invariant potential form can arise:

$$\sigma \cdot (p_i \times p_f)/m^2 \qquad \text{(IV-10)}$$

Since we wish to compare calculations made here with those of B-M,[15] this sixth form will not be included. Moreover, since it is proportional to $\frac{p^2}{m^2}$, its effects will not be noticed until high momenta, when the scattering process is assumed to be elastic and $p_i \times p_f = 0$.

This condition of elastic scattering at high intermediate momenta, may explain why the off-diagonal matrix elements of the equations coupled by the tensor force were found to be equal. For the two particle scattering problem, unitarity forces these matrix elements to be equal, but no such physical property is known to exist for the K-matrix.

Again 10 point Gaussian quadratures were used, this time in four regions:

$$r = (0 - 2),\ (2 - 10),\ (10 - 21),\ (21 - 41)\ [\text{Bev}]^{-1}\ .$$

In the last region, $u_{\ell\ell'}$ was set equal to $j_\ell \, \delta_{\ell\ell'}$. The momentum mesh used was the same as in B-M.[14]

$$k = (.1,\ .3,\ .5,\ .7,\ .9,\ 1.,\ 1.4,\ 1.8)\ p_f.$$

The Fermi momentum is given, in [Bev], as

$$p_f = 1.524/r_o \text{ if } r_o \text{ is in } [\text{Bev}]^{-1}, \text{ or}$$

$$p_f = .3006/r_o \text{ if } r_o \text{ is in fermis.}$$

$$\left\{ \left(\frac{4\pi}{3} r_o^3\right)^{-1} = \rho = \frac{N}{V} = \frac{2 \times 2 \times \frac{4\pi}{3} k_f^3}{(2\pi\hbar)^3} \right\}$$

The computation for each value of the Fermi momentum (including the exclusion principle integral) took about 7.5 seconds (30 points of r mesh), and one major iteration took about 12 minutes (2 minutes of which were used to tabulate Bessel functions, potentials, etc.).

The equations actually solved (as in the scattering problem) were for the radial wave functions $f(kr) = kr\ u(kr)$.

V. CONCLUSIONS

A. Phase Shift Analysis

The experimental data for the phase shifts to be fit was obtained from recent analyses at 25, 50, 95, 142, 210, and 310 Mev lab incident kinetic energy compiled by Noyes, et al.[22]

In order to directly compare this analysis to the data of Ball, Scotti and Wong,[17] a fit of the $^3S_1$, $^1S_0$, $^3P_0$, $^3P_1$, and $^3P_2$ was done at the six energies mentioned above. Their fixed constants were:

$m_n = .938$ (Bev) $\qquad g_\pi^2 = 13.$

$m_\pi = .140$ $\qquad g_\eta^2 = .3$

$m_\eta = .548$

Parameters

$m_v = \frac{1}{4}(m_\rho^2 + m_\omega^2 + 2m_\varphi^2) = 3.9 \qquad g_v^2 = g_{\rho_1}^2 = g_{\omega_1}^2 = g_\varphi^2 = 1.36$

$g_{\rho_2}^2 = 13.4\, g_v^2$

$g_\sigma^2 = 1.36$

Because of erratic, high-energy (higher than the experimental data) behavior in the nuclear-matter calculation, the $^1P_1$ and off-diagonal tensor potentials were cut flat at $r = 2/m_n$ instead of

$r = 1/m_n$ as the other potentials.

$$\text{A value of } \chi^2 = \sum_{\substack{i=\text{exp.}\\ \text{data}}} \left\{ \frac{\delta_i(\text{calculated}) - \delta_i(\text{experiment})}{\Delta\delta_i(\text{exp. error})} \right\}^2$$

$$= 446.5 \text{ (30 pieces of data)}$$

was obtained for the following data:

Fixed

$g_\pi^2 = 14.$ $\qquad$ $m_\pi = .135$ (Bev)

$g_\eta^2 = 14.$ $\qquad$ $m_\omega = .78$

$m_\varphi = 1.02$

$m_\eta = .548$

Parameters

$g_\sigma^2 = 3.0362$ $\qquad$ $m_\sigma = .46052$

$g_{\rho_1}^2 = 1.1$ $\qquad$ $m_\rho = .53095$

$g_\omega^2 = g_\varphi^2 = 3.02723$

$g_{\rho_2}^2 = 21.9$

m_c (highest cutoff) $= 2.32140$; $\Delta m_c = -.06752$ is the increment of m_c determining the other three cutoffs.

This fit was achieved by varying each parameter until a local minimum was reached whereupon a parabolic fit determined the 'exact' minimum of each parameter, and then this overall fit was checked by redoing the fit with each parameter.

In order to understand the meaning of this fit, as other phase shift analyses either use different data and/or rely on more primary experimental data (polarization and cross-section data), it proves enlightening to do the same calculation with two other potentials widely in use, the Breit potential (81 parameters), and the Brueckner-Gammel-Thaler (BGT) potential, both reported in BM.[15]

The BGT potential gave a χ^2 of 1596.6. The Breit potential, using a core of .506 f (iso-spin T = 0 states should have a core of .5002 f, and iso-spin T = 1 states should have a core of .5116 f), gave a χ^2 of 497.

However, the D-states are rather difficult to fit, and without any further adjustment of parameters, a χ^2 was calculated for all S, P, and D waves (66 pieces of data):

This potential (POT) gave $\chi^2 = 1483.5$

BGT gave $\chi^2 = 2540.4$

Breit (using the proper cores) gave $\chi^2 = 513.0$.

With the same parameters, a low energy analysis was done:

The scattering lengths and effective range of the $^3S_1(t)$ and $^1S_0(s)$ states were calculated using points near 5 Mev relative energy,

fitting the effective range formula:

$$k \cot \delta = -\frac{1}{a} + \frac{r_o}{2} k^2 \quad . \tag{V-1}$$

The binding energy of the deuteron can then be approximately calculated by solving

$$\gamma = \frac{1}{a_t} + \frac{r_t}{2} \gamma^2 \text{ , where}$$

$$\gamma = (-m_n \varepsilon)^{1/2}, \ \varepsilon \text{ is the binding energy,} \tag{V-2}$$

The nuclear matter code was then used to solve the deuteron problem. To do this: 1) the exclusion principle correction term was dropped, leaving the 'asymptotic' Green's functions to be the full Green's functions for the deuteron problem, and 2) the hole energy spectrum was set equal to the corresponding free energy spectrum.

The following results also agreed with results from the scattering code when the replacement $k^2 \leftrightarrow -\varepsilon m_n$ was made using the Breit potential.

A 'search' was done about the binding energies calculated above and it was affirmed that the phase shift changed sign (through $\pi/2$) in the vicinity of the correct binding energy. At this energy, the deuteron wave function was used to calculate the electric quadrupole moment of the deuteron[23] given as:

$$Q = (\Phi, S_{12} \Phi) / (\Phi, \Phi) \ , \tag{V-3}$$

Φ being the total $^3S_1$-state wave function (in the 'α-channel') - the sum of an S-state part and a D-state part: $\Phi = \frac{u}{r} + \frac{w}{r}$. Q is expressed[23] as:

$$Q = \left\{ \frac{1}{(50)^{1/2}} \int_0^\infty r^2 \, uw \, dr - \frac{1}{20} \int_0^\infty r^2 w^2 \, dr \right\} \Big/ \int_0^\infty (u^2 + w^2) \, dr \tag{V-4}$$

Table V-1

Low Energy Data

	POT	BREIT	BGT	EXPERIMENT
a_s	-94.3	-142.8	-181.2	$-120.6 \pm .357$ (Bev)$^{-1}$
r_s	15.2	14.0	13.6	12.21 ± 1.53 (Bev)$^{-1}$
a_t	20.8	25.55	21.75	$27.38 \pm .155$ (Bev)$^{-1}$
r_t	9.33	10.085	10.35	$8.71 \pm .153$ (Bev)$^{-1}$
$-\epsilon$	5.63	4.0	3.0	2.22 Mev
Q	5.75	5.41	5.46	$7.02 \pm .1$ (Bev)$^{-2}$

B. Nuclear Matter

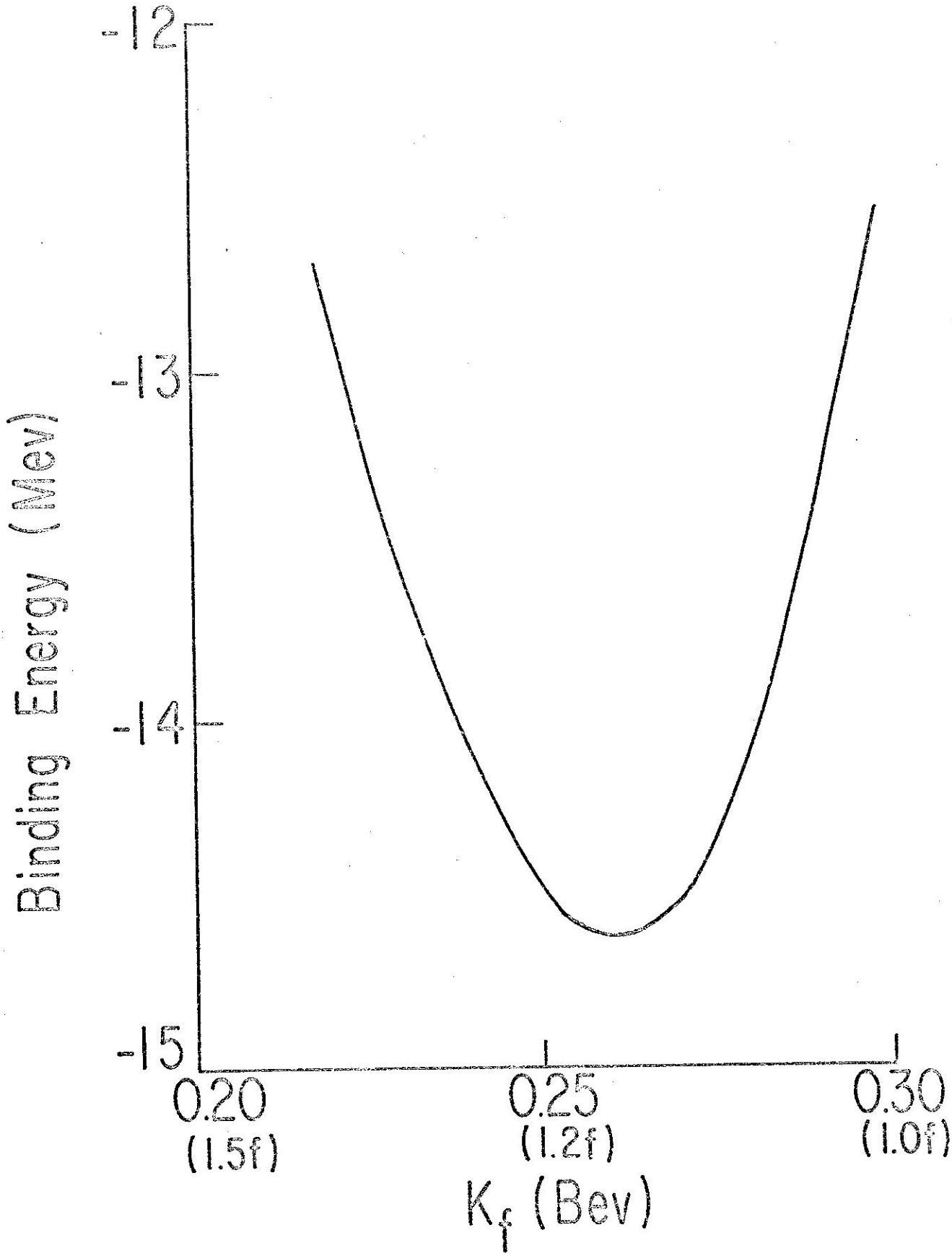
Nuclear matter calculations were done at $p_f = .22, .25, .275, .295$ (Bev), obtaining -14.6 Mev binding at $k_F = .26$ [$=> r_o = 1.15$ f]. (See Fig. 1.)

One check on the calculations was to obtain approximately equal phase shifts, for very low energies, between the nuclear matter (no exclusion principle and free energy spectrum) and the scattering computer codes. The scattering code was best checked by obtaining

FIGURE V-1

Binding Energy vs. K_f for POT $\chi^2 = 446.$

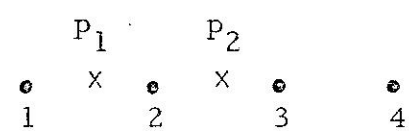
$(1f = 5.07\ [\mathrm{Bev}]^{-1})$


-12
-13
-14
-15
Binding Energy (Mev)
0.20
(1.5f)
0.25
(1.2f)
0.30
(1.0f)
K_f (Bev)

the same solutions by also solving the corresponding differential equation.

The computer code was also checked by doing a calculation at $r_o = 1.00$ using the BGT potential, thus simulating the BM[15] calculation. A binding energy of 19.3 (using the exclusion principle in just the S-states) was obtained. A repulsive contribution of 1.2 Mev is reported for the P state core contribution[8] raising this answer to 18.1, to be compared to the BM[15] result of 16.9. The additional 1.2 Mev discrepancy may in part be accounted for by the differences in mesh, and the method of interpolation used to calculate the single-particle potentials. The Gaussian-quadrature mesh gave at least five significant figure accuracy in the wave-function calculations, as demonstrated by the equality of the off-diagonal matrix elements of the coupled $^3S_1$ - $^3D_1$ system in the scattering problem.

B-M first interpolated K, then did $\int k^2 K W(k)\, dk$ to obtain the single-particle potentials. The method used here was to interpolate (k^2K), then do the integral. The most important region of k^2K, from $.3\, k_F$ to $.7\, k_F$, was found to be very closely fitted to a straight segment. The accuracy was checked by quadratically interpolating by moving the three points interpolated on, up one point, whenever the interpolated point passed the second point, i.e.:



Point p_1 uses points 1, 2, and 3 for interpolation, while point p_2 uses points 2, 3, and 4.

Then, the same procedure was done interpolating 'backwards', and averaging the results. This would average the values of p_2 calculated, by interpolating using both the points 1, 2, and 3, and the points 2, 3, and 4.

These two procedures produced less than .2 Mev difference in the binding energy, interpolating on k^2K, and .8 Mev (repulsion by not averaging) when first interpolating on K. The points were interpolated to coincide with twenty Gaussian points lying between the limits of each integral done.

As concerns the nculear matter calculation, probably the most important property of the nucleon-nucleon potential is the central/tensor ratio. This apparently non-unique ratio to be fit by the scattering data gives quite different answers for the binding energy, the larger ratio giving the greater binding.[15] On Fig. 2 is plotted the tensor force used in this calculation, the diagonal part of the kernel:

$$\frac{1}{G(r,r)} \left(V - \frac{d^2W}{dr^2} + \frac{dW}{dr}\frac{d}{dr} + \frac{Wd^2}{dr^2} \right) G(r,r) \quad , \qquad \text{(V-5)}$$

versus the tensor force of the BGT potential and the Breit potential. The large, short range repulsion of the tensor force in the scattering problem was found to be unimportant as only slight changes in the scattering problem were produced upon

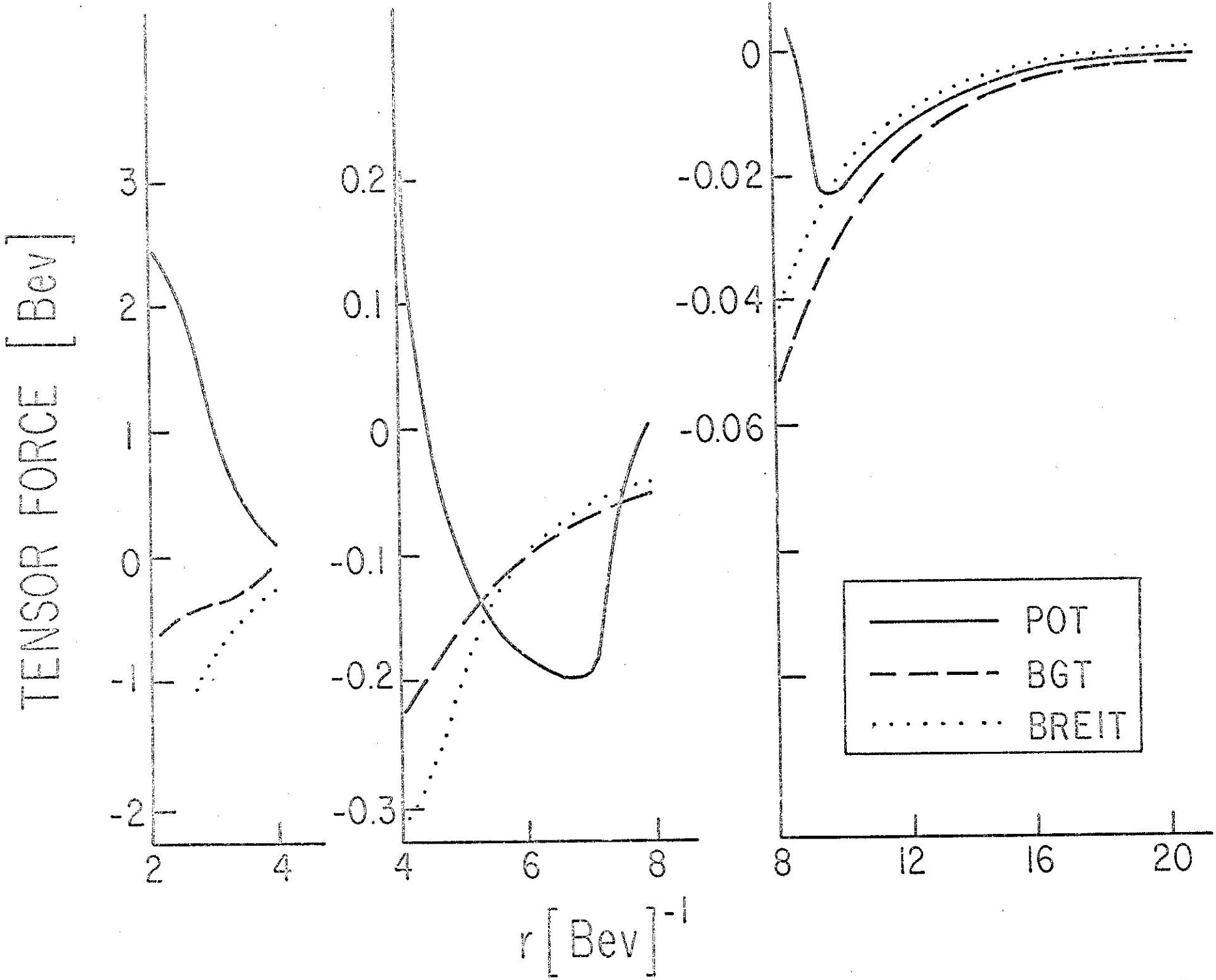
FIGURE V-2

Tensor Forces from POT, BGT, and Breit

(POT is evaluated at 95 Mev lab energy)

$(1f = 5.07\ [\mathrm{Bev}]^{-1})$

The 'bump' at $r \sim 9$ is explained in the caption of Fig. 6.


TENSOR FORCE [Bev]
3
2
1
0
-1
-2
2
4
0.2
0.1
0
-0.1
-0.2
-0.3
4
6
8
0
-0.02
-0.04
-0.06
8
12
16
20
POT
BGT
BREIT
r [Bev]$^{-1}$

erroneous inclusion of the hard core in the off-diagonal tensor potential when doing the BGT calculations. This is because the large repulsion of the central force drives the wave function → 0 in this region. The much smaller attractive part of the POT tensor force is to be noted.

Although all three potentials give substantially the same nuclear bar phase shifts for $^3S_1$, ε_1, and $^3D_1$, it is very elucidating to compare the ratio of

$$\frac{\tan^{-1}(j_o V_o \psi_1)}{\tan^{-1}(j_o V_o \psi_2)} = R_{C/T} \qquad \text{(V-6)}$$

where V_o and ψ_1 are the diagonal potential and wave function respectively, and V_T and ψ_2 are the associated coupled, off-diagonal potential and wave-function:

Table V-2

$R_{C/T}$ for Nucleon Scattering

Lab Energy (Bev)	POT	BREIT	BGT
.025	1.44/.254	-1.13/1.435	1.027/1.434
.095	1.033/-.576	-.745/1.096	.031/.78
.21	.754/-.664	-.7366/.919	-.197/.503
.31	.627/-.671	-.761/.835	-.3161/.394

Although POT gives a more 'reasonable' ratio for the $^3S_1$ matrix elements when compared to the $^1S_0$ phase shifts to which the $^3S_1$ nuclear bar phase shifts are close, this larger, central/tensor admixture can be better understood as a large

central contribution of the σ meson which is not contained in the other potentials. The argument that this contribution is necessary, by considering either the σ meson or contributions from π-π exchanges, is well formulated in SW.[1,24]

The qualitative differences due to this very large central/tensor ratio explains why these calculations give more binding in nuclear matter. In the following Table V-3, the first column gives the ratio of the two $^3S_1$ matrix elements (the sum of which contributes to the K-matrix). The denominator is the matrix element of the tensor potential. The second column gives the contribution to the K-matrix when the tensor potential is set identically to zero. The third column gives the fraction of attraction gained with $V_T = 0$ (i.e., (value with $V_T = 0$ - full value)/full value).

Table V-3. $R_{C/T}$ for K Matrix $((\text{Bev})^{-2})$

$\frac{k}{k_f}$	POT			BREIT		
	Full	$V_T = 0$	%	Full	$V_T = 0$	%
.1	$\frac{-260.0}{112.8}$	-255.6	.625	$\frac{5.6}{-103.5}$	-11.7	-.882
.5	$\frac{-176.3}{98.7}$	-167.9	1.16	$\frac{43.2}{-89.6}$	25.7	-1.92
.9	$\frac{-91.3}{76.3}$	-84.5	4.63	$\frac{56.1}{-66.5}$	36.9	-4.55

To measure the sensitivity of the binding energy and equilibrium density to these potentials, a calculation was done using parameters giving a $\chi^2 = 770$ for the $^3S_1$, $^1S_0$, $^3P_{0,1,2}$ states:

$g_\pi^2 = g_\eta^2 = 14.$	$m_\pi = .135$ (Bev)
$g_\sigma^2 = 3.05$	$m_\sigma = .45$
$g_{\rho_1}^2 = 1.1$	$m_\rho = .531$
$g_{\rho_2}^2 = 21.9$	$m_\omega = .78$
$g_\omega^2 = g_\varphi^2 = 3.$	$m_\varphi = 1.02$
	$m_\eta = .548$
$m_c = 2.3$,	$\Delta m_c = -.0667$.

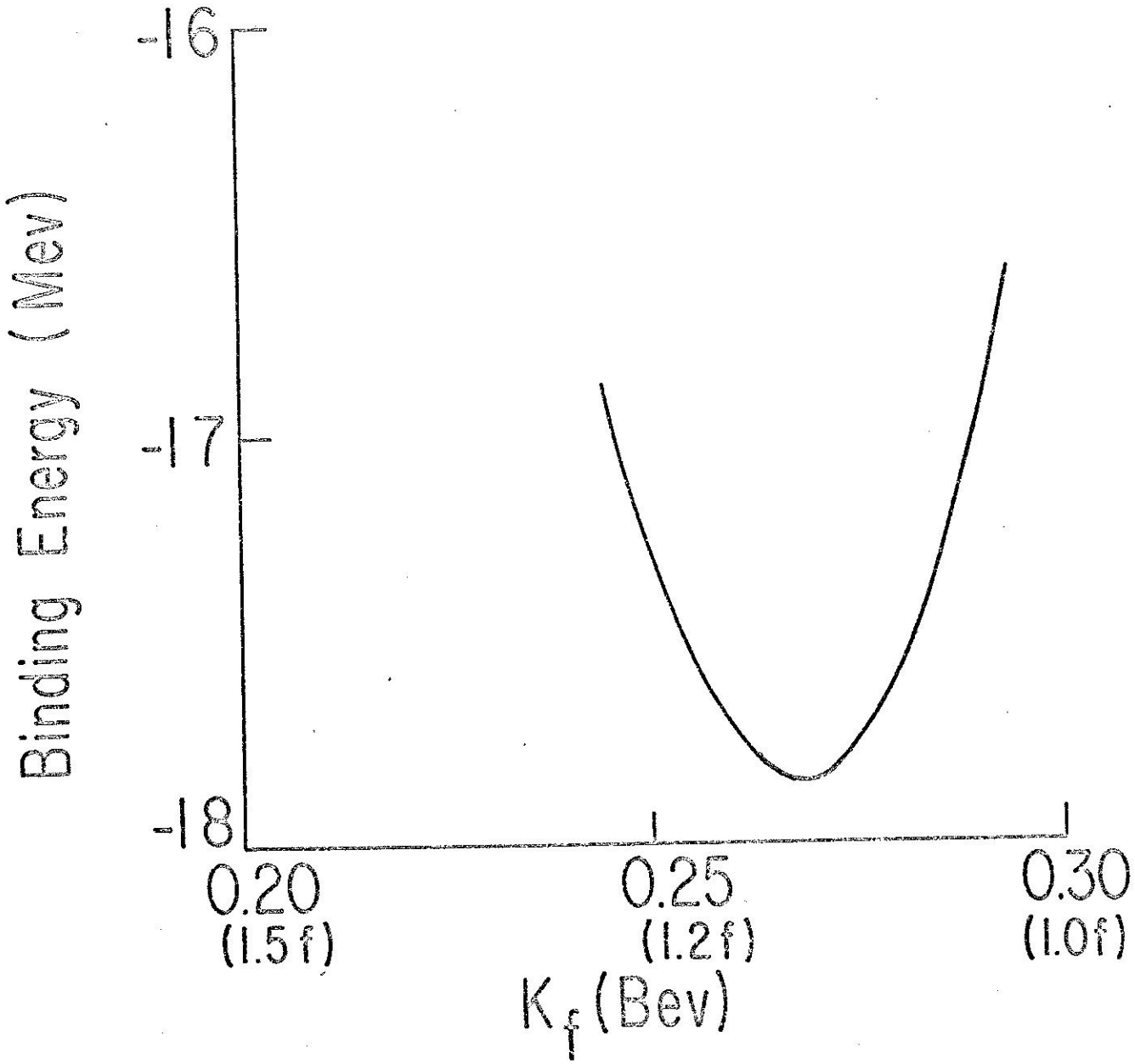
This data is not very different from that data giving $\chi^2 = 446.$, but a greater $R_{C/T}$ ratio did exist for $\chi^2 = 770.$, showing some sensitivity in even the scattering problem calculation to the effective central to tensor ratios. The result of the nuclear matter calculation was 17.8 Mev binding at a density of $k_f = .27$ ($r_o = 1.11$) for this potential (see Fig. 3).

As the D-waves were not carefully fitted to the scattering data, a comparison of D-wave contributions was made to that of the Breit potential which does fit the scattering data rather well. The Breit potential gave a D-state contribution of -7.27 Mev to the binding energy,

FIGURE V-3

Binding Energy vs. K_f for POT $\chi^2 = 770.$

($lf = 5.07\ [Bev]^{-1}$)


-16
-17
-18
Binding Energy (Mev)
0.20
(1.5 f)
0.25
(1.2 f)
0.30
(1.0 f)
K_f (Bev)

while POT gave a D-state contribution of -8.73 Mev, a difference of 1.46 Mev. Inclusion of the F, G, and H partial waves, obtained directly from the experimental phase shifts gave a repulsion of .2. If one used just Breit's D states, a binding energy of -12.9 is obtained, still 4.6 Mev lower than the Breit potential gave for the entire calculation. This result is explained by the higher central/tensor ratio, as discussed above.

Calculations at $k_f = .275$ of the $\chi^2 = 770$ potential, using the resultant self-consistent spectrums were done to test the validity of certain other approximations.

The reference spectrum approximation,[19] dropping

$$\int_0^{2.6\,k_f} dk' \left\{ \frac{f(k,k')}{E(k) - E^*(k')} - \frac{1}{E(k) - \frac{k'^2}{2m_n}} \right\} j_\ell(k'r)\, j_\ell(k'r') \quad , \tag{V-7}$$

(done for both P and D waves)

gave the same K-matrix as calculations done by including the above integral as part of the Green's function to within .1%. This is seen to be due to cancellation of

$$\int_0^{\sim k_f} dk' \, \frac{(-1.)}{E(k) - \frac{k'^2}{2m_n}} \, j_\ell \, j_\ell \, , \tag{V-8}$$

when $f(k,k')$ is $\equiv 0$, with the rest of the integral where $f(k,k') \sim 1$.

Use of the free-particle energy-spectrum for states above the Fermi surface instead of the self-consistent one in the S-state Green's function gave a binding energy of -15.47 Mev, or 1.5 Mev less binding, than the calculation using the self-consistent particle energies at $k_f = .275$.

To illustrate the importance of recognizing the momentum-dependent nature of POT, a binding energy calculation was done using the self-consistent spectrum of POT, $\chi^2 = 770$, letting $\frac{d}{dr} \to k$ (the "elastic" potential) instead of $\frac{d}{dr} \to \sqrt{2m_n E(k)}$ when operating on the asymptotic Green's functions. The result was -39 Mev, instead of -17 Mev binding due to the Green's functions' not "digging" enough into the shorter-range repulsive region of the potential $\left(\sqrt{2m_n E(k)} > k\right)$. These potentials, properly treated, are also non-local; that is, the potential is not a function of r, but rather of r and r′ when used as:

$$\psi(r) = \chi(r) + \int dr' \; X(r, r') \, G(r', r) \, \psi(r') \; ,$$

$$X(r, r') = \frac{1}{G(r, r')} \left(V - \frac{d^2 W}{2dr'^2} - \frac{dW}{dr'} \frac{d}{dr'} - W \frac{d^2}{dr'^2} \right) G(r, r') \; . \tag{V-9}$$

The non-locality can usually be completely shouldered onto the $\frac{d}{dr}$ term as $\frac{d^2}{dr^2}$ can be reexpressed as $\left(\frac{L^2}{r^2} \pm \alpha^2\right)$ operating on simple Green's functions by use of the homogeneous Bessel's equation. α corresponds to the proper momentum conjugate to the radial parameter in the Green's function, and the + or - sign is to be taken depending on whether a bound-state or scattering problem, respectively, is to be done. The $\frac{d}{dr}$ term cannot be so treated.

The local part of the potential for the $^1S_0$ state is:

$$V_\ell(r') - \frac{d^2 W_\ell(r')}{m_n^2\, dr'^2} + W_\ell(r') \frac{k^2}{m_n^2} , \qquad \text{(V-10)}$$

and to this is added the non-local part (for the scattering calculation):

$$-\frac{dW(r')}{dr'} \frac{k}{m_n^2} \frac{d}{d(kr')} (\sin(kr_<) \cos(kr_<)) / (\sin(kr_<) \cos(kr_>)) \qquad \text{(V-11)}$$

where $r_<$ is the lesser of r and r', and $r_>$ is the greater. On Fig. 4 is plotted the local potential above and $\frac{-1}{m_n} \frac{dW(r')}{dr'}$ for $k^2 = .0445$.

(In the nuclear matter calculation, there is some slight additional non-locality from the $\frac{d^2}{dr^2}$ term operating on that part of the Green's function containing the exclusion principle correction.)

Exhibited on Fig. 5 is the diagonal 'energy'-dependent $^3S_1$ potential in nuclear matter at a value of $k/k_f = .1$ ($k_f = .275$). On Fig. 5 is plotted similar data for the off-diagonal tensor force. The diagonal part includes local and non-local contributions. The $^3S_1$ was calculated with the exclusion principle correction.

That the S-states are so sensitive to the shape of this potential in the small r region is, of course, no less physical than the sensitivity of similar calculations to the "hard-core" radius, as was demonstrated when using the Breit potential.

As can be seen from Eq. (V-11), the contribution of the diagonal, non-local potential to the local potential is:

$$- \frac{dW(r)}{dr} \frac{k}{m_n^2} \cot(2\,kr)\,, \quad \left(= - \frac{dW}{dr} \frac{\alpha}{m_n^2} \coth(2\,\alpha r) \text{ for a bound state calculation} \right) \tag{V-12}$$

which for small values of kr is:

$$- \frac{dW(r)}{dr} \frac{1}{2\,m_n^2\,r} \quad . \tag{V-13}$$

This is obtained from contributions at points r' approaching r from above and below (notice - this largest contribution to the potential, for small r, is energy-independent; outside this range $W'' \to 0$ fastest and its k^2 contribution is thus diminished)

$$\begin{aligned} \psi(kr) = \chi(kr) + \cos(kr) \int_0^r \sin(kr')\, \overleftarrow{K}(r', \nabla_{r'})\, \psi(kr')\, dr' \\ + \sin(kr) \int_r^\infty \cos(kr')\, \overleftarrow{K}(r', \nabla_{r'})\, \psi(kr')\, dr' \,. \end{aligned} \tag{V-14}$$

However, as $r \to 0$, the only contribution to Eq. (V-11) comes from the second integral, and in contrast to Eq. (V-12), one obtains:

$$\begin{aligned} &+ \frac{dW(r)}{dr} \frac{k}{m_n^2} \tan(kr), \text{ which for small kr:} \\ &= \frac{dW(r)}{dr} \frac{k^2 r}{m_n^2}\,, \quad \left(= \frac{dW}{dr} \frac{\alpha}{m_n^2} \text{ for the bound state} \right) \end{aligned} \tag{V-15}$$

and does not diverge for small r. This is necessary to satisfy the S-state boundary conditions. (When multiplied by the Green's function, this $r = 0$ contribution is negligible.) Unlike the "δ-function" behavior

of the $\frac{d^2}{dr^2}$ term, this "step-function" behavior of the $\frac{d}{dr}$ term cannot be explicitly taken out from the integral, and one must include this somewhat discontinuous behavior in order to carry out the mathematics of this physical model. As can be seen from Fig. V-4, the $\frac{dW}{dr}$ term quickly becomes insignificant when compared to the local part of the potential, as r increases, so that these sensitive, non-local effects are only important for the S-states.

FIGURE V-4

Local and Non-local Contributions to the $^1S_0$-state Potential,

$k^2 = .0445$ (95 Mev incident lab energy)

$(1f = 5.07\ [\mathrm{Bev}]^{-1})$

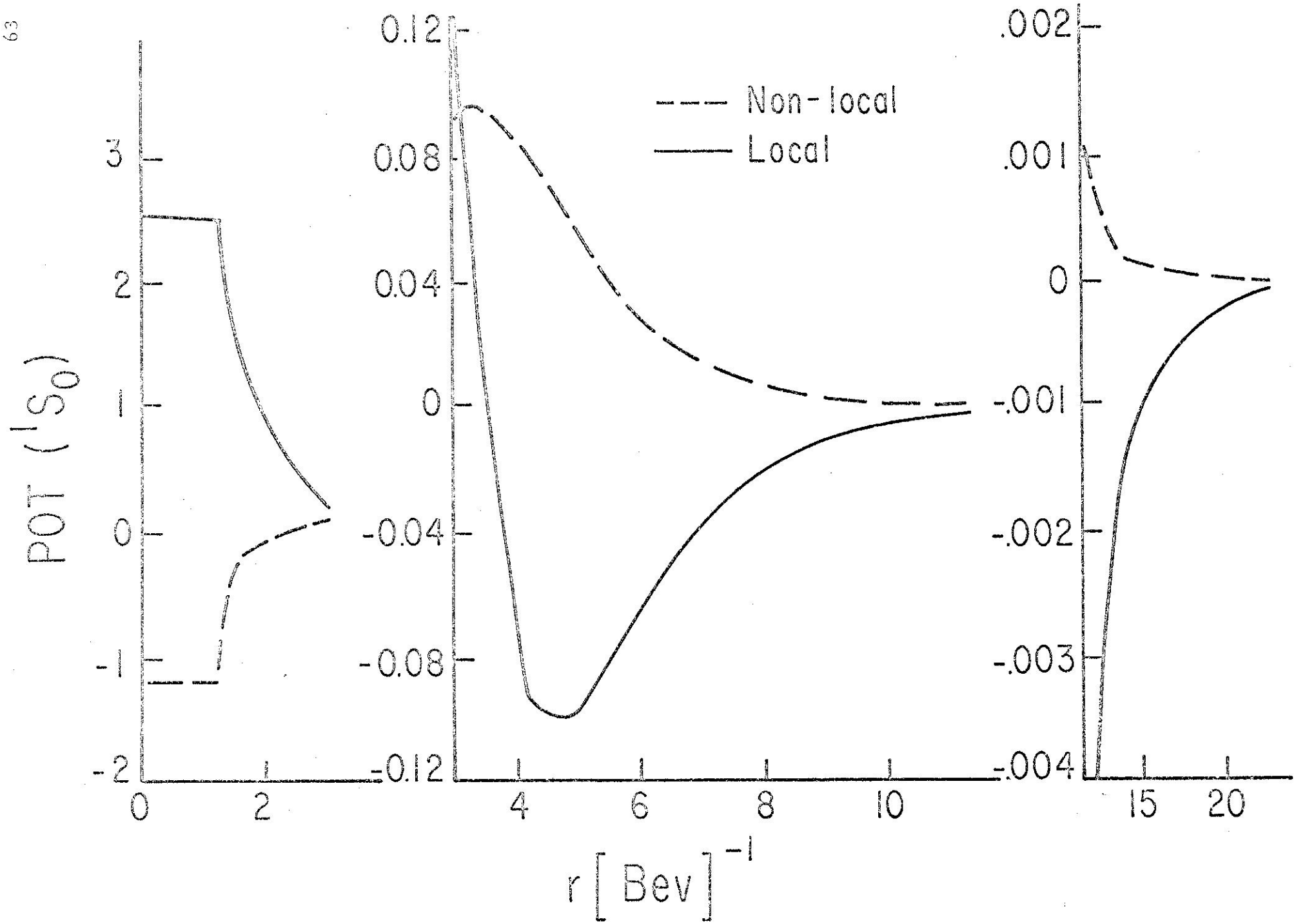

POT ($^1S_0$)
3
2
1
0
-1
-2
0
2
0.12
0.08
0.04
0
-0.04
-0.08
-0.12
4
6
8
10
.002
.001
0
-.001
-.002
-.003
-.004
15
20
Non-local
Local
r [Bev]$^{-1}$

FIGURE V-5

$^3S_1$ POT for $\frac{k}{k_f} = .1$, $k_f = .275$

$(1f = 5.07\ [\mathrm{Bev}]^{-1})$

As mentioned in the text, these potentials, with proper treatment of both momentum and non-locality possess very little energy-dependence. In fact, the potentials in Figs. 5 and 6 vary only a few percent, for $r \leqslant 2$, when calculated at $\frac{k}{k_f} = .9$, $k_f = .275$

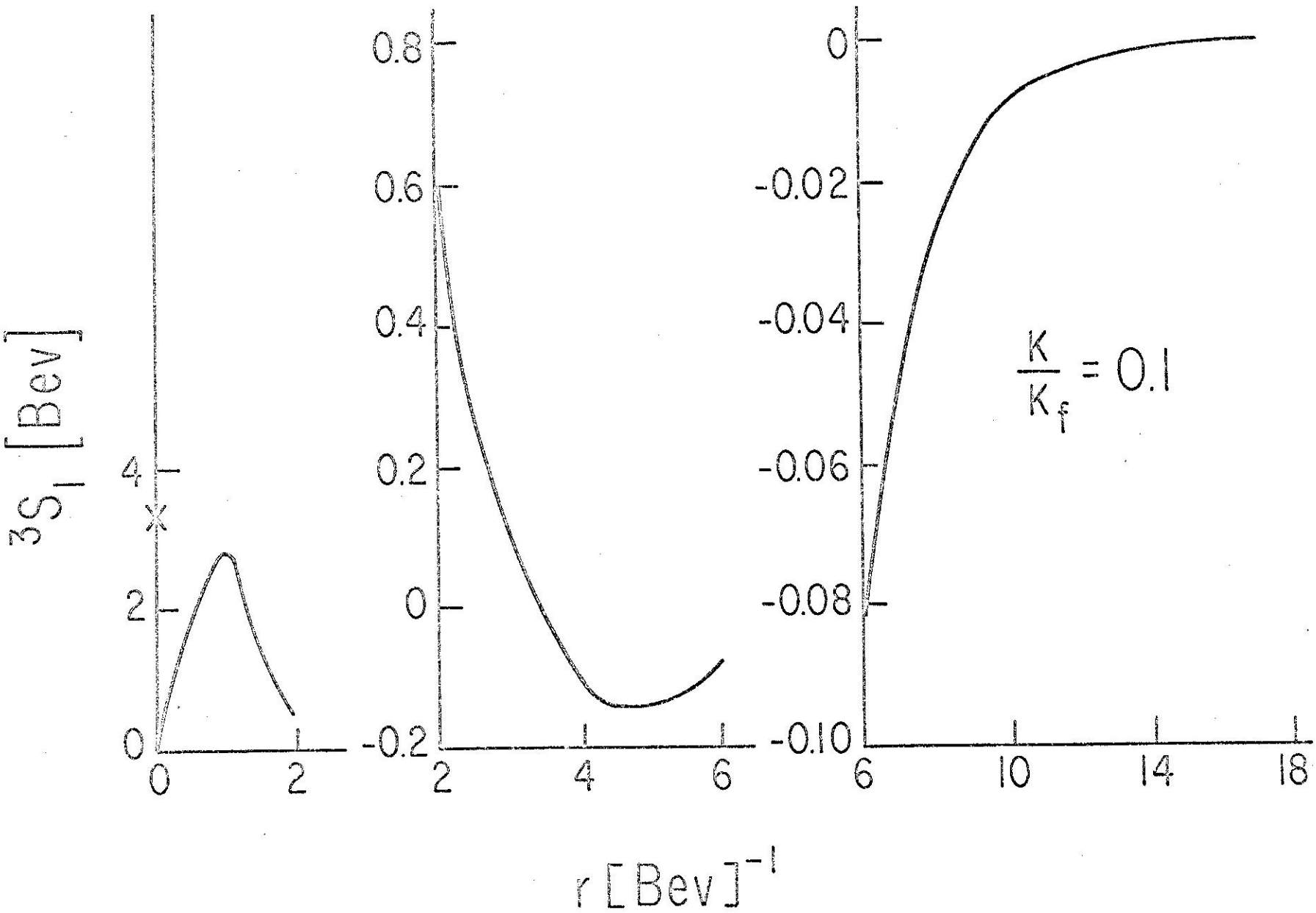
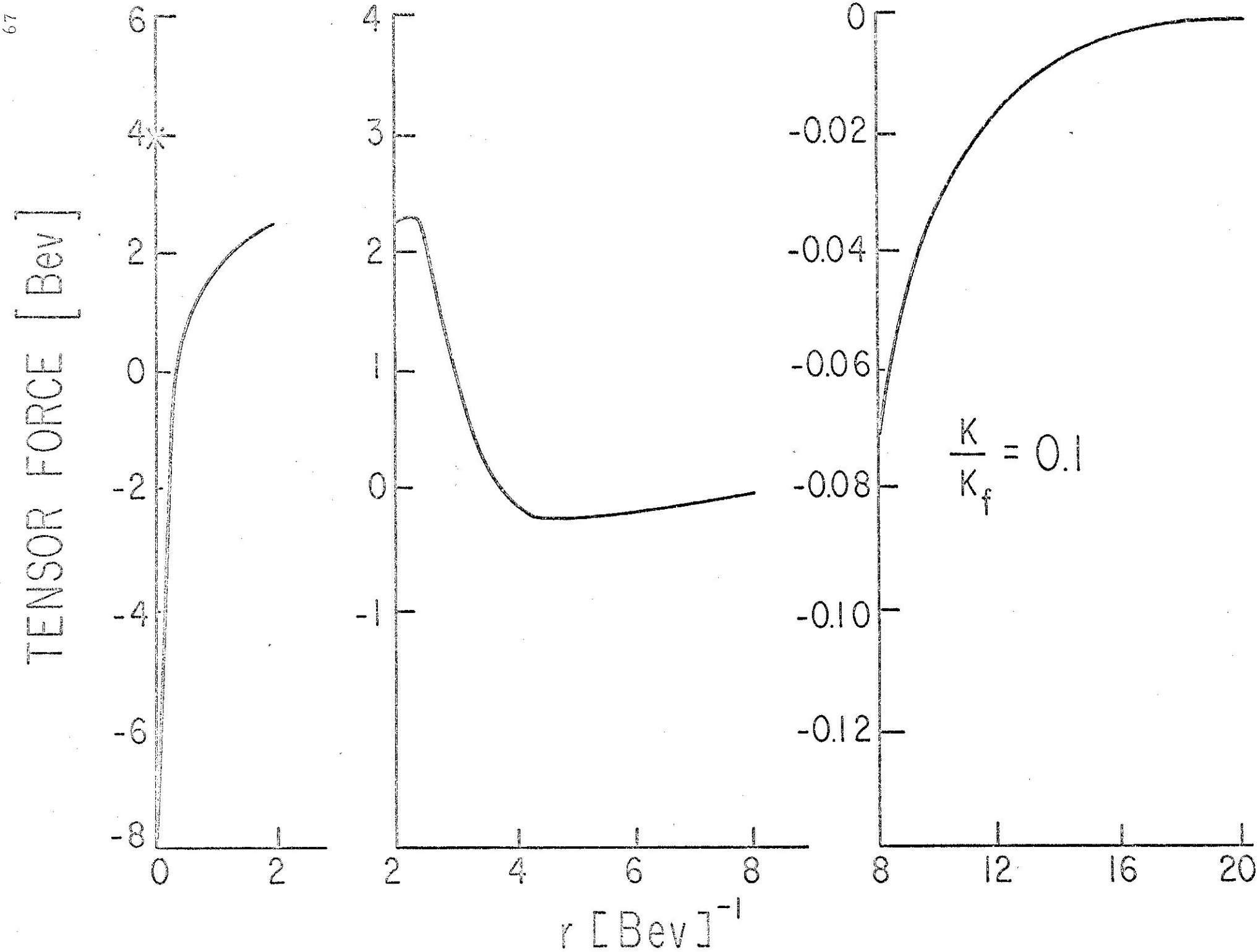

$^3S_1$ [Bev]
4
2
0
0
2
0.8
0.6
0.4
0.2
0
-0.2
2
4
6
0
-0.02
-0.04
-0.06
-0.08
-0.10
6
10
14
18
$\frac{K}{K_f} = 0.1$
r [Bev]$^{-1}$

FIGURE V-6

Tensor POT for $\frac{k}{k_f} = .1$, $k_f = .275$

$(1f = 5.07\ [\mathrm{Bev}]^{-1})$

Comparison between this Fig. 6 and Fig. 2 illustrates the important, different contributions, between the scattering and bound-state problems, from the non-local potential. (Notice that $kr = \pi \simeq (.0445)^{1/2} \times 9$ explains the anomalous bump in Fig. 2).


TENSOR FORCE [Bev]
6
4
2
0
-2
-4
-6
-8
0
2
4
3
2
1
0
-1
2
4
6
8
0
-0.02
-0.04
-0.06
-0.08
-0.10
-0.12
8
12
16
20
K/K_f = 0.1
r [Bev]^-1

If one treats the momentum-dependence correctly, there does exist an approximation for obtaining a local, energy-dependent potential by approximating $\frac{d}{dr} \rightarrow \alpha$. This is seen to be true for both the scattering and the nuclear matter calculations. Only the S-states, (especially the $^3S_1$ state) in each case, really suffer from this approximation. Of course, the larger α, the worse the approximation. Notice should be taken however, that because of such error in the S-states the overall calculations are qualitatively quite poor, illustrating the importance of non-locality in the 'core' (small r) region. (See tables 4 and 5.)

In summation, one concludes that this one-meson exchange potential has, in most cases quantitatively, and in all cases qualitatively explained the non-relativistic observables of Nuclear Physics - with only eight (8) parameters, which themselves are reasonable physical quantities!

Table V-4
Non-locality in Nucleon Scattering

Phase Shifts

	Lab. Energy (Bev)	With G′, $\chi^2 = 1438.5$	With $\alpha G(\alpha = k)$ $\chi^2 = 93,560$
$^1S_0$	.025	.895	.973
	.142	.249	.335
	.31	-.192	-.06
$^3P_1$	.025	-.093	-.093
	.142	-.268	-.276
	.31	-.321	-.331
$^3D_2$	.025	.064	.065
	.142	.566	.552
	.31	.712	.654
$^3S_1$	.025	1.48/-.432	-1.56/1.45
$R_{C/T}$	.142	.888/-.572	.718/-.192
	.31	.625/-.668	.448/-.363

Table V-5
Non-locality in Nuclear Matter ($k_f = .275$)

K matrix ($[\text{Bev}]^{-2}$)

	k/k_f	With G′	With $\alpha G(\alpha = \sqrt{2m_n E(k)})$
$^1S_0$	.3	-99.5	-95.9
	.7	-38.9	-36.8
$^3P_1$	.3	10.75	10.78
	.7	15.64	15.74
$^3D_2$	.3	-4.26	-4.25
	.7	-17.8	-17.5
$^3S_1$	.3	-222.8 / 107.3	-244.3 / 175.9
$R_{C/T}$	.7	-130.9 / 88.2	-135.8 / 124.1
Binding Energy		-14.48 Mev	+8.48

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APPENDIX A: POTENTIAL (POT)

For each meson considered,[11] five F functions (F_c, F_σ, F_T, F_{LS}, $F_{\sigma p}$) will be given. For the first four, the $\frac{1}{E}$ phase-space factor is to be later expanded to $\frac{1}{m}\left(1 - \frac{k^2}{2m^2}\right)$. The fifth, $F_{\sigma p}$, is already multiplied by $\frac{1}{m^2}\ \sigma_1 \cdot k\ \sigma_2 \cdot k$ and proportional to k^2/m^2, so $\frac{1}{E}$ is replaced by $\frac{1}{m}$. Each time a factor Δ^{2n} appears, it is understood to be regularized by the methods given in Section II. Integrals over momentum transfer space, for

$$F_c,\ F_\sigma,\ F_{\sigma \cdot p} \text{ give, with } F = \frac{+\, g_\mu^2\, m^2}{E(\mu^2 + \Delta^2)}, \tag{A-1}$$

a term $-\frac{g_\mu^2\, e^{-\mu r}}{r}$.

Integrals of $F_{LS} = \frac{+\, g_\mu^2}{E(\mu^2 + \Delta^2)}$, give a term

$$-\frac{g_\mu^2\, e^{-\mu r}}{r} \left(\frac{1}{\mu^2\, r^2} + \frac{1}{\mu r}\right) \frac{\mu^2}{m^2}. \tag{A-2}$$

Integrals of $F_T = \frac{g_\mu^2}{E(\mu^2 + \Delta^2)}$, give a term

$$-\frac{g^2\, e^{-\mu r}}{r} \left(\frac{3}{\mu^2\, r^2} + \frac{3}{\mu r} + 1\right) \frac{\mu^2}{m^2}. \tag{A-3}$$

The F's given are the I = 0 (isospin = 0) potentials due to the exchange of I = 0 particles. The other three possibilities are obtained from the I-spin crossing matrix:

I-spin of Potential = (crossing matrix) × I-spin of Particle,

or

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{A-4}$$

Pseudoscalar Meson (π, η)

$$\begin{aligned} F_c &= 0 \\ F_\sigma &= \frac{g^2}{12E} \frac{\Delta^2}{\mu^2 + \Delta^2} \\ F_T &= \frac{g^2}{12E} \frac{1}{\mu^2 + \Delta^2} \\ F_{LS} &= 0 \\ F_{\sigma p} &= 0 \end{aligned} \tag{A-5}$$

Scalar Meson (σ)

$$F_c = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(m^2 + \frac{\Delta^2}{4} + \frac{\Delta^4}{64m^2} - \frac{\Delta^2 k^2}{16m^2} - \frac{\Delta^4 k^2}{128m^4} \right)$$

$$F_\sigma = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^4}{64m^2} - \frac{\Delta^2 k^2}{24m^2} - \frac{\Delta^4 k^2}{128m^4} \right)$$

(Equation A-6 continued on next page.)

$$F_T = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(- \frac{k^2}{48m^2} \right)$$

A-6)

$$F_{LS} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{2} + \frac{\Delta^2}{16m^2} - \frac{k^2}{8m^2} - \frac{\Delta^2 k^2}{32m^4} \right)$$

$$F_{\sigma p} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{16} \right)$$

Vector Meson (ρ, ω, φ)

Vector coupling

$$F_c = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(- m^2 + \frac{\Delta^2}{2} - \frac{\Delta^4}{64m^2} - 2k^2 + \frac{\Delta^2 k^2}{16m^2} + \frac{\Delta^4 k^2}{128m^4} \right)$$

$$F_\sigma = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{6} - \frac{\Delta^4}{64m^2} + \frac{\Delta^2 k^2}{24m^2} + \frac{\Delta^4 k^2}{128m^4} \right)$$

$$F_T = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{12} + \frac{k^2}{48m^2} \right) \qquad \text{(A-7)}$$

$$F_{LS} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{3}{2} - \frac{\Delta^2}{16m^2} + \frac{k^2}{8m^2} + \frac{\Delta^2 k^2}{32m^4} \right)$$

$$F_{\sigma p} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(- \frac{\Delta^2}{16} \right)$$

Tensor coupling (for the ρ only)

$$F_c = \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(- \frac{\Delta^4}{16m^2} + \frac{\Delta^6}{256m^4} - \frac{\Delta^4 k^2}{64m^4} - \frac{\Delta^6 k^2}{512m^6} \right)$$

$$F_\sigma = \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{6} - \frac{\Delta^4}{8m^2} + \frac{\Delta^6}{256m^4} + \frac{\Delta^2 k^2}{3m^2} - \frac{\Delta^4 k^2}{96m^4} - \frac{\Delta^6 k^2}{512m^6} \right) \qquad \text{(A-8)}$$

$$F_T = \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{12} + \frac{k^2}{6m^2} - \frac{\Delta^2 k^2}{192m^4} \right)$$

$$F_{LS} = \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(- \frac{3\Delta^2}{8m^2} + \frac{\Delta^4}{64m^4} + \frac{\Delta^2 k^2}{32m^4} - \frac{\Delta^4 k^2}{128m^6} \right)$$

$$F_{\sigma p} = \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(- \frac{\Delta^2}{2} + \frac{\Delta^4}{64m^2} \right)$$

Mixed vector-tensor coupling

$$F_c = \frac{g g_2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{2} - \frac{\Delta^4}{8m^2} + \frac{\Delta^2 k^2}{2m^2} + \frac{\Delta^4 k^2}{32m^4} \right)$$

(Equation A-9 continued on next page.)

$$F_{\sigma} = \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{3} - \frac{\Delta^4}{8m^2} + \frac{\Delta^2 k^2}{3m^2} + \frac{\Delta^4 k^2}{32m^4} \right)$$

$$F_T = \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{6} + \frac{k^2}{6m^2} \right) \qquad \text{(A-9)}$$

$$F_{LS} = \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(2 - \frac{\Delta^2}{2m^2} + \frac{k^2}{m^2} + \frac{\Delta^2 k^2}{8m^4} \right)$$

$$F_{\sigma p} = \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(- \frac{\Delta^2}{2} \right)$$

APPENDIX B: CORRECT USE OF FIFTH $\sigma \cdot p$ POTENTIAL

[See page]

To express $\sigma \cdot p \sigma \cdot p$ as an eigenoperator of J^2, L^2, S^2, write:

$$p = \nabla / i, \tag{B-1}$$

$$\nabla = \hat{r}(\hat{r} \cdot \nabla) - \frac{\hat{r} \times (r \times \nabla)}{r}$$

$$= \hat{r} \frac{\partial}{\partial r} - i \frac{\hat{r} \times L}{r}, \text{ and} \tag{B-2}$$

$$\sigma \cdot \nabla = \sigma \cdot \hat{r} \frac{\partial}{\partial r} - i \frac{\sigma \cdot r \times L}{r^2}$$

$$= \sigma \cdot \hat{r} \frac{\partial}{\partial r} - \frac{\sigma \cdot r \, \sigma \cdot L}{r^2} \tag{B-3}$$

The Dirac identity gives:

$$\sigma \cdot r \, \sigma \cdot L = r \cdot L + i \sigma \cdot r \times L = i \sigma \cdot r \times L, \tag{B-4}$$

$$\sigma \cdot \nabla = \sigma \cdot \hat{r} \left[\frac{d}{dr} - \frac{\sigma \cdot L}{r} \right]. \tag{B-5}$$

Using

$$- \frac{S \cdot L}{r} \frac{d}{dr} - \frac{d}{dr} \frac{S \cdot L}{r} = - \frac{2S \cdot L}{r} \frac{d}{dr} + \frac{1}{r^2} S \cdot L, \tag{B-6}$$

one obtains:

$$\sigma_1 \cdot \nabla \sigma_2 \cdot \nabla = \sigma_1 \cdot \hat{r}\, \sigma_2 \cdot \hat{r} \left[\frac{d^2}{dr^2} - \frac{(\sigma_1 + \sigma_2) \cdot L}{r} \frac{d}{dr} \right.$$

$$\left. + \frac{\sigma_1 \cdot L\, \sigma_2 \cdot L}{r^2} + \frac{(\sigma_1 + \sigma_2) \cdot L}{r^2} \right] . \qquad \text{(B-7)}$$

Using

$$2S = \sigma_1 + \sigma_2 , \qquad \text{(B-8)}$$

$$4(S \cdot L)^2 = \sigma_1 \cdot L \sigma_1 \cdot L + \sigma_2 \cdot L \sigma_2 \cdot L + 2 \sigma_1 \cdot L \sigma_2 \cdot L \qquad \text{(B-9)}$$

and

$$\sigma \cdot L \sigma \cdot L = L \cdot L + i \sigma \cdot L \times L = L^2 - \sigma \cdot L , \qquad \text{(B-10)}$$

$$\sigma_1 \cdot L \sigma_2 \cdot L = 2(S \cdot L)^2 - L^2 + S \cdot L , \qquad \text{(B-11)}$$

yields

$$\sigma_1 \cdot \nabla \sigma_2 \cdot \nabla = \sigma_1 \cdot \hat{r}\, \sigma_2 \cdot \hat{r} \left[\frac{d^2}{dr^2} - \frac{2S \cdot L}{r} \frac{d}{dr} + \frac{2(S \cdot L)^2 - L^2 + 2S \cdot L}{r^2} \right] \qquad \text{(B-12)}$$

Finally:

$$\sigma \cdot p \sigma \cdot p = -\frac{1}{3} (S_{12} + \sigma_1 \cdot \sigma_2) \left[\frac{d^2}{dr^2} - \frac{2S \cdot L}{r} \frac{d}{dr} + \frac{2(S \cdot L)^2 - L^2 + 2S \cdot L}{r^2} \right] \qquad \text{(B-13)}$$

where

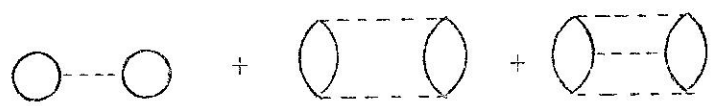
$$S_{12} = 3\, \sigma_1 \cdot \hat{r}\, \sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2$$

is the tensor operator.

APPENDIX C: DERIVATION OF NUCLEAR MATTER EQUATIONS

According to the rules set down in Section I, K. A. Brueckner's theory selectively sums the perturbation series. This method rules out the possibility of calculating collective properties such as superconductivity. However, other investigations show this state, if it exists, to be close to the ground state calculated here.[18]

First the 'ladder sum' is calculated:



(C-1)

i.e.,

$$t_{ij;k\ell} = V_{ij;k\ell} + \sum_{mn} V_{ij;mn} \frac{(1-n_m)(1-n_n)}{\epsilon_k + \epsilon_\ell - \epsilon_m - \epsilon_n} V_{mn;k\ell} + \dots ,$$

V is the two-body potential, and ϵ is the kinetic energy.

In terms of this t-matrix, the energy shift (expectation value of the Hamiltonian) in Brueckner-Goldstone theory is given as:[8]

$$\Delta E = \frac{1}{2} \sum_{k\ell} t_{k\ell;k\ell} n_k n_\ell - \frac{1}{2} \sum_{\substack{k\ell \\ mn}} t_{mn;k} \frac{n_k n_\ell (1-n_m)(1-n_n)}{(\epsilon_k + \epsilon_\ell - \epsilon_m - \epsilon_n)^2}$$

$$\times t_{mn;(k\ell)} \times \left\{ \sum_j \left(t'_{ij;jk} + t'_{j\ell;(j\ell)} - t'_{jm;(jm)} \right.\right.$$

$$\left.\left. - t'_{jn;(jn)} \right) n_j \right\} + (\text{hole - hole} + 3 \text{ body clusters of}$$

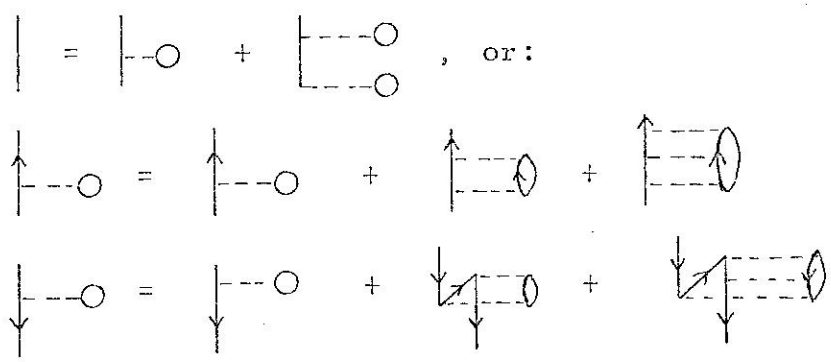
$$\text{order } t^3 + \dots) , \qquad \text{(C-2)}$$

where

$$t'_{jk;(jk)} = V_{jk;jk} + \sum_{mn} V_{jk;mn} \frac{(1-n_m)(1-n_m)}{(\epsilon_j + \epsilon_k - \epsilon_m - \epsilon_n - \delta E)} t'_{mn;(jk)}$$

(δE is later approximated, and set equal to $E(k_f) - E(0)$).

Next, self-energy inserts are made:



(C-3)

This is done by redefining the propagator:

$$\frac{1}{\epsilon_k + \epsilon_\ell - \epsilon_m - \epsilon_n} \longrightarrow \frac{1}{\omega_k' + \omega_\ell' - \omega_m' - \omega_n'} \tag{C-4}$$

where $\omega_k' = \epsilon_k + \sum_\ell t_{k\ell;(k\ell)} n_k n_\ell$.

Next, one defines a new reaction matrix, K:

$$K_{ij;k\ell} = V_{ij;k\ell} + \sum_{mn} V_{ij;mn} \frac{(1-n_m)(1-n_n)}{E_k' + E_\ell' - E_m' - E_n'} K_{mn;k\ell}$$

where

$$E_k' = \epsilon_k + \sum_j K_{jk;(jk)} n_j = \epsilon_k + \mathcal{V}_k ,$$

and the energy shift becomes:

$$\Delta E = \frac{1}{2} \sum_{k\ell} \mathbb{K}_{k\ell;(k\ell)} \, n_k \, n_\ell + 0 \, (\mathbb{K}^3) \tag{C-6}$$

However, the energy shift for particles is taken in the presence of excited states which produces a shift δE in propagators defining their intermediate states:

$$\mathbb{K}^*_{ij;k\ell} = V_{ij;k\ell} + \sum_{mn} V_{ij;mn} \frac{(1 - n_m)(1 - n_n)}{E'_k + E'_\ell - E'_m - E'_n - \delta E} \mathbb{K}^*_{mn;k\ell} . \tag{C-7}$$

Mention should be made that another summation has been done by Bethe[19] of the $O(\mathbb{K}^3)$ and higher terms in a density expansion. The first term of this new sum contains three hole lines (as compared to two hole lines in $\mathbb{K}$). However, it has recently shown[20] that these terms taken together with the high partial waves of the two body $\mathbb{K}$ matrix gives a negligible contribution to the total energy in high intermediate states.

The approximations necessary for a partial wave reduction[21] are:

1) angle average of energies

2) angle average of the exclusion principle.

A further approximation[8] was introduced by Masterson to average the center of mass momenta to facilitate computations.

The final equations are given in Section IV.