

Nuclear Forces*†

LESTER INGBER‡

Department of Physics and Institute for Radiation Physics and Aerodynamics, University of California, San Diego, La Jolla, California 92038

(Received 21 August 1967; revised manuscript received 27 May 1968)

One-meson-exchange Feynman diagrams are nonrelativistically reduced and unitarized via Schrödinger's equation. Properties of nucleon-nucleon scattering are calculated at incident laboratory energies of 25–310 MeV. Bound-state properties of the deuteron and of nuclear matter are also calculated. Mesons included are the π , η , σ , ρ , ω , and φ . Very good over-all agreement with the experimental data is obtained. Important features of this "potential" include its momentum dependence, properly treated, and the contribution of the σ "meson," which qualitatively changes the central/tensor force ratio from that of previous phenomenological potentials.

I. INTRODUCTION

PREVIOUS theoretical and experimental studies demonstrate that forces between two nucleons arise from their mutual interaction via mesons. Because of theoretical difficulties, however, it has been traditional to describe this interaction by an arbitrary function with parameters fitted to experimental data. The problem to date has been to maximally utilize the basic theory of meson exchanges to develop a consistent formalism to deal with nuclear forces in a variety of physical situations including scattering and bound-states problems.

In the elastic scattering region below inelastic thresholds, about 350-MeV incident lab energy, important progress has been made by Scotti and Wong.¹ They postulated forces due to meson exchange via a Lagrangian interaction, and (essentially) unitarized this interaction Lagrangian via methods of dispersion relations. Despite some uncertainties associated with descriptions of the mesons (Regge cutoffs, etc.) and with dispersion theory itself (subtraction, unitarization, etc.), a good fit to the experimental data was achieved within the elastic scattering region (0–400-MeV incident lab energy).

Within this energy region, it may be possible to describe an interacting system of nucleons (scattering and bound states) with a formalism consistent with Schrödinger's equation. This would enable this interaction to be used with more confidence in other nuclear-physics calculations, especially in those problems like nuclear matter, which are constructed within a Schrödinger framework. Besides introducing a new functional form of the nucleon-nucleon interaction, maximally utilizing meson-exchange information, this momentum-dependent potential can be used more consistently than previous potentials to describe the basic nucleon-nucleon interaction in different physics problems: Other poten-

tials are "fit" to the scattering data using different functional forms in different partial waves. As the nucleons really interact via the entire interaction simultaneously present in all partial waves, it seems to be more consistent to use, in other physics problems like nuclear matter, an interaction with parameters common to the entire interaction (like coupling constants and meson masses), rather than parameters defined only in individual partial waves. It should be noted that investigators working with other potentials have consistently expressed concern when applying their potentials to nuclear matter.²

The method is straightforward: A nonrelativistic reduction of Lagrangian field equations describing free nucleons yields the Schrödinger equation upon minimization of the action. It is now proposed to make a similar nonrelativistic reduction of the total Lagrangian (including interactions via meson exchange) to the same "order" (nonrelativistic reduction of Feynman amplitudes) and attempt to "correctly" solve the resulting equations. Some deficiencies of the theory are ignored, such as those associated with divergences of the ρ vector meson; higher nonrelativistic corrections are also not considered. In fact, no attempt has been made to show that this theory is a proper nonrelativistic reduction of any relativistic theory.

For the nuclear-matter calculation, the Brueckner-Masterson formalism (many-body Schrödinger theory) is used.³

II. POTENTIALS DERIVED FROM LAGRANGIAN

The following Lagrangians representing the interaction of nucleons with other particles or resonances will be considered:

π meson (pseudoscalar in coordinate space and isovector in charge space):

$$\mathcal{L}_\pi = (4\pi)^{1/2} g_\pi \bar{\psi} \gamma_5 \tau \cdot \varphi_\pi \psi, \quad (2.1)$$

where g_π is the coupling strength of the meson field φ_π to the nucleon fields ψ and $\bar{\psi}$.

² K. Lassila, M. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).

³ K. S. Masterson, thesis, University of California, San Diego, 1963 (unpublished).

* Submitted in partial fulfillment for Ph.D. requirements at University of California, San Diego.

† Supported by the U. S. Atomic Energy Commission.

‡ Present address: Lawrence Radiation Laboratory, University of California, Berkeley, Calif.

¹ A. Scotti and D. Wong, Phys. Rev. **138**, B145 (1965), referred to as SW in the text.

η meson (pseudoscalar in coordinate space and isoscalar in charge space):

$$\mathcal{L}_\eta = (4\pi)^{1/2} g_\eta \bar{\psi} \gamma_5 \varphi_\eta \psi. \quad (2.2)$$

σ meson (scalar in both coordinate and charge spaces): The σ is not established to be a *bona fide* meson.⁴ It will be used here to approximately represent a strong π - π *S*-wave resonance.¹ The "parameters" g_σ and μ_σ (mass) will hopefully simulate two-pion-exchange effects:

$$\mathcal{L}_\sigma = (4\pi)^{1/2} g_\sigma \bar{\psi} \varphi_\sigma \psi. \quad (2.3)$$

ρ meson (vector in both coordinate and charge spaces):

$$\mathcal{L}_\rho = i(4\pi)^{1/2} (g_{\rho 1} + g_{\rho 2}) \bar{\psi} \gamma_\nu \boldsymbol{\tau} \cdot \boldsymbol{\varphi}_\rho \psi - (4\pi)^{1/2} (g_{\rho 2}/2m_\rho) (\mathbf{q} + \mathbf{q}') \cdot \bar{\psi} \boldsymbol{\tau} \cdot \boldsymbol{\varphi}_\rho \psi. \quad (2.4)$$

The tensor coupling constant (or anomalous magnetic-current term) $g_{\rho 2}$ is also included as suggested by the experimentally determined magnetic moment form factor of the nucleon.¹

ω and φ mesons (vectors in coordinate space and scalars in charge space):

$$\mathcal{L}_{\omega(\varphi)} = i(4\pi)^{1/2} g_{\omega(\varphi)} \bar{\psi} \gamma_\nu \varphi_{\omega(\varphi)} \psi. \quad (2.5)$$

[These five mesons (including the tensor coupling of the ρ) also exhaust the five possibilities of Dirac γ matrices that can describe the various meson exchanges.]

A spinor representation⁵ for ψ is then picked:

$$\psi = e^{iq \cdot x} u_\Lambda^{(1,2)}(\mathbf{q}), \quad (2.6)$$

where Λ takes the value $+1$ or -1 corresponding to a positive or negative energy state, respectively, and 1 or 2 designates positive or negative helicity, i.e.,

$$u_\Lambda^1(\mathbf{q}) = \left(\frac{\epsilon + m}{2\epsilon} \right)^{1/2} \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{q}}{\epsilon + m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix}, \quad (2.7)$$

$$u_\Lambda^2(\mathbf{q}) = \left(\frac{\epsilon + m}{2\epsilon} \right)^{1/2} \begin{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{q}}{\epsilon + m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix};$$

$\epsilon = \Delta E = \Delta(p^2 + m^2)^{1/2}$, and m designates the rest mass of the nucleon, taken to be the same for protons and neutrons, and $p^2 = \hbar^2 q^2 = q^2 (\hbar = 1)$ is the square of the three-momentum (\mathbf{p} will be the operator ∇ in Schrödinger's equation). The inner column matrices in Eq.

⁴ W. D. Walker, Rev. Mod. Phys. 39, 693 (1967).

⁵ S. S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Co., New York, 1961).

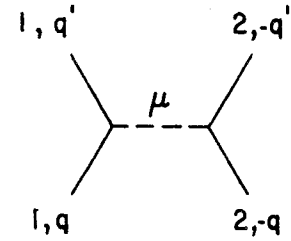


FIG. 1. Feynman diagram of one-meson-exchange contribution to nuclear-nuclear scattering.

(2.7) designates the spin states:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

has just a positive helicity component.

For example, the "potential," before a nonrelativistic reduction, for the π meson (with mass μ) is (see Fig. 1)

$$M(\mathbf{q}, \mathbf{q}') \sim -\frac{(E' + m)(E + m)}{4EE'} \boldsymbol{\sigma}_1 \cdot (\mathbf{q}' - \mathbf{q}) \times \frac{g^2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2}{\Delta^2 + \mu^2} \boldsymbol{\sigma}_2 \cdot (\mathbf{q}' - \mathbf{q}), \quad (2.8)$$

where $\Delta^2 = (\mathbf{q} - \mathbf{q}')^2$ is the square of the momentum transferred via the π meson. [For inelastic scattering, the "propagator" is

$$\frac{1}{\Delta^2 - (E' - E)^2 + \mu^2}, \text{ or } q'^2 \neq q^2.$$

For nuclear matter, where inelastic scattering can take place, one can only hope that $|E' - E| \ll \mu^2$. Also, see the comment about $\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_f - \mathbf{p}_i) \boldsymbol{\sigma}_2 \cdot (\mathbf{p}_f - \mathbf{p}_i)/m^2$ made at Eq. (4.10) in the section on nuclear matter.]

There are five independent amplitudes for each meson exchanged, depending on the final and initial z components of the total spin of the nucleons:

Partial-wave amplitude	Corresponding helicity amplitude	
	In	Out
singlet $J=L$	(1) $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$
triplet $J=L$	(2) $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$
coupled $J=L \pm 1$	(3)	$\downarrow\downarrow$
	(4)	$\uparrow\uparrow$
	(5)	$\downarrow\uparrow$
	(6)	$\uparrow\downarrow$
(triplet)	(6)	$(\uparrow\uparrow \quad \downarrow\downarrow)$

[amplitude (6) is equal to the amplitude for elastic scattering, (5) due to invariance under time reversal].

For elastic scattering, there are conveniently also five independent "Wigner amplitudes"⁶ in coordinate space that can be constructed out of bilinear combinations of the spin and momenta (to order p^2). It is "wise" to pick

$$1, \sigma_1 \cdot \sigma_2, (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S}, (3\sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2) \equiv S_{12}; \quad (2.9)$$

$$\sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p} / m^2,$$

where $\mathbf{p} = \nabla_r / i$. Thus, the potential in coordinate space can be obtained from the amplitude in momentum space, Eq. (2.8):

$$W = W_1 + W_{\sigma_1 \sigma_2} \sigma_1 \cdot \sigma_2 + W_{S_{12}} S_{12} + W_{LS} \mathbf{L} \cdot \mathbf{S} + \sigma_1 \cdot \mathbf{p} W_{\sigma_1 \sigma_2} \sigma_2 \cdot \mathbf{p}, \quad (2.10)$$

defined by

$$W(r, p^2) = \int d\Delta e^{i\Delta \cdot r} M(\Delta, p^2). \quad (2.10')$$

To facilitate the calculation, an intermediate set of amplitudes in momentum space is picked:

$$f_\lambda(\Delta, p^2) = \sum_{\substack{i=1, \sigma_1 \sigma_2, S_{12}, \\ LS, \sigma_1 \mathbf{p}}} a_i^\lambda(\Delta) M_i(\Delta, p^2), \quad (2.11)$$

where f_λ is, for example, one of the five invariant helicity amplitudes which have been found from the amplitudes for meson exchange. These five equations are then inverted to find

$$M_i(\Delta, p^2) = \sum [a^{-1}]_\lambda^i f_\lambda(\Delta, p^2). \quad (2.12)$$

Any five independent amplitudes could be picked for the f_λ . Wong⁷ chose these helicity amplitudes because of their nice analyticity properties. For example, for f_λ one could pick in momentum space

$$1, \sigma_1 \cdot \sigma_2, (\sigma_1 + \sigma_2) \cdot (\mathbf{q} \times \mathbf{q}');$$

$$-\frac{1}{2} i \sigma_1 \cdot (\mathbf{q} - \mathbf{q}') \sigma_2 \cdot (\mathbf{q} - \mathbf{q}'), \quad (2.9')$$

$$-\frac{1}{2} i \sigma_1 \cdot (\mathbf{q} + \mathbf{q}') \sigma_2 \cdot (\mathbf{q} + \mathbf{q}').$$

This set of projection amplitudes lends itself quite easily to reduction of the Feynman amplitudes with the nucleon spinors expressed in the helicity representation.⁵ They also have the advantage that the corresponding "a" matrix in Eq. (2.12) is quite easy to calculate, and so are their Fourier transforms, Eq. (2.10').

Finally, one takes spherical Bessel transforms, which arise from the angular reduction of the Fourier transforms, to find $W_i(r, p^2)$. The five Wigner operators are eigenoperators of eigenfunctions of the total angular momentum ($L+S=J$), and by expressing the potential by means of this set, the W_i lend themselves, trivially, to a partial-wave reduction. For this purpose $\sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p}$

is given in a more useful form in Appendix B.

$$M_{(1; \sigma_1 \sigma_2; \sigma_1 \mathbf{p})} = -m \int W_{(1; \dots)} j_0(\Delta r) r^2 dr, \quad (2.13)$$

j_0 being the spherical Bessel function of zeroth order.

$$M_{S_{12}} = (-S_{12}(\Delta))(-m) \int W_{S_{12}}(r) j_2(\Delta r) r^2 dr, \quad (2.14)$$

$$M_{LS} = \mathbf{S} \cdot \mathbf{n} i p^2 \frac{\sin \theta}{2\Delta} (-m) \int W_{LS}(r) j_1(\Delta r) r^3 dr, \quad (2.15)$$

where

$$\mathbf{n} = \mathbf{q} \times \mathbf{q}' / |\mathbf{q} \times \mathbf{q}'|$$

and

$$|\mathbf{q} \times \mathbf{q}'| = p^2 \sin \theta, \quad p^2 \cos \theta = \mathbf{q} \cdot \mathbf{q}'.$$

The nonrelativistic reduction of $W(r, p^2)$, to get a potential $V(r, p^2)$, is done by expanding all powers of

$$\frac{E}{M} = \frac{(p^2 + m^2)^{1/2}}{m} \sim 1 + \frac{p^2}{2m^2} + \frac{p^4}{6m^4} + \frac{p^6}{24m^6} + \dots, \quad (2.16)$$

and by keeping factors like $\cos \theta = 1 - \Delta^2 / 2p^2$ and $1/p^2$ which occur in the a_λ^i . A phase-space factor of $m / (m^2 + p^2)^{1/2}$ is also expanded to $1 - p^2 / 2m^2$, allowing local potentials to be defined.⁷ Then all powers higher than p^2/m^2 are dropped. This reduction is thus valid for $p^2/m^2 \ll 1$. The velocity dependence is seen to arise from factors of $(p^2 + m^2)^{1/2}$ and products of $\sigma \cdot \mathbf{q}$ (combinations of which are determined by the type of meson being exchanged) from the nucleon spinors.

To remove the δ -function singularities at $r=0$, due to infinities in momentum-transfer space

$$\left[\int \frac{\Delta^n j_n(\Delta r)}{\Delta^2 + \mu^2} d\Delta \quad \text{diverges for } n \geq 0 \right],$$

cutoffs in the form of products of propagators $[\prod_i (\Lambda_i^2 - \mu^2) / (\Lambda_i^2 + \Delta^2)]$, reexpressed as sums of propagators after partial fractionation] multiplying the integrands of integrals over momentum-transfer space were introduced. $\prod_i (\Lambda_i^2 - \mu^2)$ was chosen as a factor to enable g^2 for each meson to have the usual meaning [$V \sim g^2 \times e^{-\mu r} / r$] in the asymptotic region. Four Λ_i 's were needed to take care of the highest power of Δ found in the "induced-tensor" term of the ρ meson. (All amplitudes were consistently multiplied by the same cutoff factor.)

Inspection of the various Lagrangians and of the nucleon spinors reveals that, because the spinors are well-behaved functions of q and q' , only the ρ -meson derivative coupling term should have any momentum divergences. However, when expanding the spinor normalization factors,

$$\frac{\Delta^2}{p^2 + m^2} \sim \frac{\Delta^2}{m^2} \left[1 - \frac{p^2}{m^2} \right], \quad (2.17)$$

⁶ J. L. Gammel and R. M. Thaler, Progr. Elem. Particle Cosmic Ray Phys. 5, 99 (1960).

⁷ D. Wong, Nucl. Phys. 55, 212 (1963).

such divergences "appear." The "masses" used to regularize these divergences were, however, larger than the nucleon mass which imposed the harshest cutoff restrictions as explained below.

It should be clear that these cutoffs do nothing more than mask a complete ignorance of the potential region within $r \leq 1/m$. Any reasonable cutoff which will ensure an analytic form of the potential throughout coordinate space will serve this purpose. This analytic behavior of the potential is necessary to correctly carry out the simple mathematics performed later in connection with the momentum-dependent potential. These cutoffs are physically justified only if the contribution to matrix elements from regions within $r \leq 1/m$ are independent of the form of the cutoff. Therefore, it is of interest to do more calculations with other soft-core and/or momentum-dependent potentials.⁸

To decrease the singular nature of the potential at small distances of r , especially the $1/r^3$ behavior of the $\mathbf{L} \cdot \mathbf{S}$ and S_{12} terms, all potentials in coordinate space were multiplied by $(1 - e^{-rm})^3$. As these momentum-dependent potentials are expansions in p^2/m^2 , and as the highest mass of mesons considered is on the order of 1 BeV/ c^2 [φ meson], they should not be considered as having any meaning within a distance of $\sim 1/m$, or 0.2 F. Also, the potential within $r < 1/m$ was set equal to its value at $r = 1/m$ to aid the numerical solution of the phase shifts. For the same "physical" reasons, the cutoffs used in the integrals over momentum transfer should be on the order of 1 BeV/ c^2 (but greater than the mass of the φ meson). This also influenced the choice of meshes which was determined by ensuring that, within each interval at least several points spanned both the Compton wavelength of the most predominant meson in that interval, as well as the conjugate variable to r in the Green's functions of Eqs. (3.7'), (4.6), and (5.1').

Mention should be made that the cutoffs recommended in Ref. 7

$$\left(\int_0^\infty d\Delta \sim \int_m^\infty d\Delta \right)$$

give rise to unphysical oscillatory potentials dying off as inverse powers of r , for large values of r .

A natural way to symmetrize the momentum-dependent p^2 terms arising from the energy expansion is

$$\text{Symm } (p^2 v) = \frac{1}{2}(p^2 v + v p^2). \quad (2.18)$$

However, the $\sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p}$ terms could be written as

$$\frac{1}{2}(\sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p} + v \sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p}) \quad (2.19a)$$

⁸ Many other investigators have worked on the problem of finding suitable potentials to describe both the scattering and nuclear-matter problems. For other approaches, see J. Goto and S. Machida, *Progr. Theoret. Phys. (Kyoto)* **25**, 64 (1960); N. Hoshizaki and S. Machida, *ibid.* **24**, 1325 (1960); N. Hoshizaki, I. Lin, and S. Machida, *ibid.* **26**, 680 (1961).

(again symmetrical in spins 1 and 2) or as

$$\frac{1}{2}(\sigma_1 \cdot \mathbf{p} v \sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \mathbf{p} v \sigma_1 \cdot \mathbf{p}). \quad (2.19b)$$

Form (b) was chosen, but since a complete set of amplitudes is used, either form may be used.

The $1/r^3$ singularities are quite unphysical. The attractive tensor potential used without a cutoff would predict no lowest eigenvalue for the two-nucleon system (i.e., a 3P_1 bound state with infinite binding energy) which, of course, can exist as a deuteron. This cutoff in a purely mechanical problem was first noticed by Hans Bethe⁹ in 1940 as being necessary to calculate properties of the deuteron with the two-Yukawa potentials used at that time.

III. SCATTERING MATRIX

Schrödinger's equation

$$((-\hbar^2/m)\nabla^2 + v(\mathbf{r}))\psi_{E^+}(\mathbf{r}) = E\psi_{E^+}(\mathbf{r}), \quad E = k^2/m \quad (3.1)$$

can be recast into an integral equation

$$\psi^+(\mathbf{r}) = \varphi(\mathbf{r}) + (m/\hbar^2) \int G^+(\mathbf{r}-\mathbf{r}')v(\mathbf{r}')\psi^+(\mathbf{r}')d\mathbf{r}', \quad (3.2)$$

where $\varphi = e^{i\mathbf{k}\cdot\mathbf{r}}$ is a plane wave and $G^+ = -(1/4\pi|\mathbf{r}-\mathbf{r}'|) \times e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$ is the Green's function for the problem assuming outgoing spherical waves as a boundary condition. (k^2 is the total-energy eigenvalue and should not be confused with q^2 in Sec. II. The origins of the coordinate-space momentum dependence in the potential, as deduced from the total Lagrangian, clearly dictate that $p^2 = -\nabla^2$, and not $p^2 = k^2$.)

The procedure to solve Schrödinger's equation with a potential $v(\mathbf{r})$,

$$(\nabla^2 + k^2)\psi^+(\mathbf{kr}) = mv(\mathbf{r})\psi^+(\mathbf{kr}), \quad (3.3)$$

is to first solve the homogeneous problem

$$(\nabla^2 + k^2)\varphi(\mathbf{kr}) = 0, \quad (3.4)$$

then to solve for the Green's function

$$(\nabla_{r^2} + k^2)G_k^+(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \quad (3.5)$$

where G_k^+ is determined by a sum over the eigenfunctions determined by the solution of the homogeneous equation. This procedure leads to the wave equation

$$\psi^+(\mathbf{kr}) = \varphi(\mathbf{kr}) + m \int d\mathbf{r}' G_k^+(\mathbf{r},\mathbf{r}')v(\mathbf{r}')\psi^+(\mathbf{kr}'), \quad (3.6)$$

which expresses ψ^+ as a sum of the homogeneous and inhomogeneous solutions of the differential equation.

For $v(\mathbf{r})$ rotationally invariant, a partial-wave reduc-

⁹ H. A. Bethe, *Phys. Rev.* **57**, 260 (1940).

tion may be made to find δ_i ;

$$\begin{aligned} \psi_i^+(r) &= j_i(r) + m \int G_i^+(r, r') v_i(r') \psi_i^+(r') dr', \\ (\psi^+(k\mathbf{r}) &= \sum_l (2l+1) (i)^l P_l(\hat{k} \cdot \hat{r}) \psi_i^+(kr)), \end{aligned} \tag{3.7}$$

giving $\delta_i(k)$, the phase shift, determined by

$$e^{i\delta_i} \sin \delta_i = -m \int j_i(r) v_i(r) \psi_i^+(r) dr. \tag{3.8}$$

Similarly,

$$\psi_i(r) = j_i + m \int G_i(r, r') v_i(r') \psi_i(r') dr' \tag{3.7'}$$

gives

$$\tan \delta_i = -m \int j_i v_i \psi_i dr, \tag{3.8'}$$

where G_i is the Green's function with standing-wave boundary conditions. Neglecting spin, for readability, the standing-wave solution is

$$\lim_{r \rightarrow \infty} \psi_i(kr) = j_i(kr) - \eta_i(kr) \tan \delta_i(k). \tag{3.9}$$

j_l and η_l are spherical Bessel and Neumann functions,¹⁰ respectively. This is the result obtained with $G_i(r', r) \sim j_i(r_<) \eta_i(r_>)$, $r_<$ and $r_>$ being the lesser and greater of r and r' , respectively.

However, the potential derived in Sec. II is momentum-dependent,

$$(\nabla^2 + k^2) \psi_k(\mathbf{r}) = mV(\mathbf{r}, \nabla) \psi_k(\mathbf{r}), \tag{3.10}$$

and to be useful for nuclear-matter calculations a correct integral-equation representation must be found.

If one "retransforms" the differential equation to resemble Schrödinger's equation with a pure radial function as a potential, the potential becomes a function of k ,¹¹ thereby destroying the eigenvalue properties of the integral equation. Alternative forms leave the homogeneous equation and the Green's-function equation with a complicated function multiplying the Laplacian operator which destroys the usefulness of the simple Green's-function solutions and physical intuition gained when using a static potential. For example, the differential equation

$$[\nabla^2 + k^2 - (x(r) + y(r)d/dr + z(r)d^2/dr^2)] \psi = 0 \tag{3.11}$$

can certainly be solved by numerical means (for x , y , and z well behaved). However, previous authors have, by setting $y=0$, recast this equation into the form

$$[(1-z)\nabla^2 + k^2 - x - \frac{1}{2}z''] \psi = 0; \tag{3.12}$$

¹⁰ L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., New York, 1955).

¹¹ A. N. Green, *Nucl. Phys.* 33, 218 (1961). In this paper the potential was used incorrectly in an integral equation to discuss nuclear matter.

and some have rewritten Eq. (3.12), disregarding the possibility of division by zero, first as

$$\left(\nabla^2 + \frac{k^2 - x - \frac{1}{2}z''}{1-z} \right) \psi = 0 \tag{3.13}$$

and finally as

$$\left(\nabla^2 + \frac{k^2z - x - \frac{1}{2}z''}{1-z} + k^2 \right) \psi = 0, \tag{3.14}$$

giving an "energy-dependent" potential, $v(\mathbf{r}, k^2) = (x + \frac{1}{2}z'' - k^2z)/(1-z)$.

It is felt that a better procedure to solve

$$(\nabla^2 + k^2) \psi(k\mathbf{r}) = mV(\mathbf{r}, \nabla) \psi(k\mathbf{r}), \tag{3.15}$$

without modifying the simple boundary conditions of Eq. (3.17), is to first solve

$$(\nabla^2 + k^2) \chi(k\mathbf{r}) = 0, \tag{3.16}$$

then solve

$$(\nabla_r + k^2) G_k(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{3.17}$$

One thereby obtains

$$\psi(k\mathbf{r}) = \chi(k\mathbf{r}) + m \int dr' G_k(\mathbf{r}, \mathbf{r}') \{ V(\mathbf{r}', \vec{\nabla}_{r'}) \psi(k\mathbf{r}') \}, \tag{3.18}$$

the arrow over ∇ indicating the direction of operation. After a partial-wave reduction,

$$\psi_i = j_i + m \int dr' j_i(r_<) \eta_i(r_>) \{ V_i(r', \vec{\nabla}_{r'}) \psi_i(kr') \}, \tag{3.19}$$

($r_<$ is the lesser of r and r' , and $r_>$ the greater), which gives a formula for the phase shift,

$$\tan \delta_i(k) = -m \int dr j_i(kr) \{ V_i(r, \vec{\nabla}) \psi_i(kr) \}, \tag{3.20}$$

$$mV_i(r, \vec{\nabla}) = x_i(r) + y_i(r)d/dr + z_i(r)d^2/dr^2$$

for the potential forms derived in Sec. II. Similar coupled equations hold for the triplet $J=L\pm 1$ partial waves [see Eq. (4.6)].

Integration by parts, for a potential well-behaved at the origin (say, as a constant), and decreasing exponentially at infinity, yields

$$\begin{aligned} (1-Z_i) \psi_i &= j_i + \int dr \\ &\times \psi_i \{ X_i + Y_i d/dr + Z_i d^2/dr^2 \} G_i, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \tan \delta_i &= - \int dr \psi_i \{ X_i + Y_i d/dr + Z_i d^2/dr^2 \} j_i, \\ &= m \int dr \psi_i V_i(\vec{\nabla}, r) j_i. \end{aligned} \tag{3.22}$$

The singularity from d^2G/dr^2 has been explicitly taken out and added to the left-hand side of Eq. (3.21).

Y_i contains a d/dr term from the $\sigma \cdot p$ amplitude which previously could not be properly used in calculations using the "transformed equations" [Eq. (3.13)] which is the probable reason many previous calculations just used four of the five possible independent potential forms. Since $G_i(r, r')$ and $j_i(r)$, and their derivatives, are simple functions of sines and cosines, these equations are no harder to solve than those using a potential depending solely on a radial parameter.

This was checked numerically by solving the 1S_0 -state equation at 95-MeV incident lab energy, first by a Runge-Kutta method for the differential equation, and then by using matrix inversion for the integral equation. Only the potential used as defined here gave the correct wave function and phase shifts when comparison between the two solutions were made. Comparisons were made with the formulation of Green¹¹ in the energy range of 25–310-MeV incident lab energy. Disagreements of 1–12% were found, the larger occurring at higher energies. This served as both a check on the accuracy of the integral equation (meshes, etc.) and also as a verification of the importance of solving the correct eigenvalue problem presented above. In light of this, previous calculations involving energy-dependent potentials and/or energy-dependent boundary conditions should be reexamined as to their regions of validity.

The integrals were done using three regions of 10-point Gaussian quadratures:

- (1) $r = 0-3$ (BeV/ $\hbar c$)⁻¹,
- (2) $r = 3-12$ (BeV/ $\hbar c$)⁻¹,
- (3) $r = 12-30$ (BeV/ $\hbar c$)⁻¹.

The conversion factor from fermis to (BeV/ $\hbar c$)⁻¹ is 1 F = 5.0686 (BeV/ $\hbar c$)⁻¹. The mass of the nucleon, m , is taken to be the averaged masses of the proton and neutron:

$$m = 0.93886 \text{ BeV}/c^2.$$

To solve

$$\psi_r = \chi_r + \int dr' K_{r,r'} \psi_{r'} \simeq \chi_r + \sum_{r'} K_{r,r'} \psi_{r'} \equiv \chi_r + (K\psi)_r, \quad (3.23)$$

matrix inversion was performed:

$$\psi = [1 - K]^{-1} \chi. \quad (3.24)$$

IV. NUCLEAR MATTER

The two properties sought in a nuclear-matter calculation are the volume term of the Bethe-Weizsacker semiempirical mass formula, and the uniform saturation density of large nuclei.³ The experimental binding energy is taken to be -15.5 ± 2.0 MeV. The mean spacing

between nuclei in a saturated large nucleus is taken to be 1.12 ± 0.02 F.

The calculations of nuclear matter, as performed by BM,¹² were done for various potentials. It therefore seemed more fruitful to do a similar many-body calculation in order to directly compare the effects of the functional forms of those potentials to this work.⁸ It was also hoped that the well-behaved potentials used would make the approximations used in BM more palatable.

The theory developed by Brueckner¹² describes the interactions of nucleons with energies below the surface of the Fermi sea. The surface, or boundary, of a Fermi sea of fermions (k_F , in terms of the momentum) is defined as the highest occupied energy level at zero temperature. The Brueckner theory describes these interactions via "particles" above the Fermi sea, corresponding to repeated interactions ("ladder sum") of each pair; and the theory also takes into account the forward scattering ("self-energy") of the excited "particles" with the unexcited "holes" of the nuclear medium, to infinite order.

Since BM did not calculate the correct observed binding energy of nuclear matter, continuing research has been concerned both with the examination of Brueckner theory itself, as well as with the "input" to the theory, which is the nucleon-nucleon interaction. The former avenue of research has led to the approximate summing of three-body scattering corrections. As calculated by Bethe and by Day, these corrections turn out to be rather small.¹² The latter avenue of research has led to the present work.

First to be calculated are Green's functions

$$G_i(r, r') = \frac{1}{\pi} \int_0^\infty \frac{k''^2 dk'' j_i(k''r) j_i(k''r') f(P, k'')}{[E(k) - E^*(k'')]}. \quad (4.1)$$

for on-energy shell propagation, and with $E(k) - E^*(k'') - \Delta E$ as a denominator for off-energy shell propagation (see Eq. 4.8b); $E(k) = E(k_F)$ for $k \geq k_F$ due to averaging of the center-of-mass momenta. This last approximation adds about 1-MeV binding to the binding energy.¹² ΔE is an approximate mean excitation energy = $E(k_F) - E(0)$. P is an average total momentum

$$P^2 \equiv P_{av}^2 = \frac{12}{5} k_F^2 \left(1 - \frac{k}{k_F}\right) \left(1 + \frac{k}{2k_F} + \frac{k^2}{6k_F^2}\right) / \left(1 + \frac{k}{k_F}\right), \quad (4.2)$$

for $k < k_F$

and $P^2 \equiv P_{av}^2 = 0$, for $k \geq k_F$.

¹² K. A. Brueckner and K. S. Masterson, Phys. Rev. **128**, 2267 (1962), referred to as BM in the text. This paper contains references to previous preliminary works on "Brueckner" theory. For a current analysis of Brueckner-Goldstone theory and associated problems in nuclear matter, see the articles by B. D. Day [Rev. Mod. Phys. **39**, 719 (1967)], R. Rajaraman and H. A. Bethe [*ibid.* **39**, 745 (1967)], and B. H. Brandow [*ibid.* **39**, 771 (1967)].

$f(P, k'')$ is the angular average of the Pauli exclusion-principle projection operator:

$$\begin{aligned} f(P, k'') &= 0, & (k'' + P^2/4)^{1/2} < k_F \\ &= 1, & (k'' - P/2) > k_F \\ &= \frac{k'' + P^2/4 - k_F^2}{k'' P}, & \text{otherwise.} \end{aligned} \quad (4.3)$$

The integral from zero to infinity was calculated as¹³

$$\begin{aligned} &\frac{1}{\pi} \int_0^{2.6k_F} dk'' k''^2 j_i(k''r) j_i(k''r') \\ &\times \left\{ \frac{f(P, k'')}{E(k) - E^*(k'')} \frac{1}{[E(k) - k''^2/2m]} \right\} \\ &+ \frac{1}{\pi} \int_0^\infty \frac{dk'' k''^2 j_i(k''r) j_i(k''r')}{[E(k) - k''^2/2m]}, \end{aligned} \quad (4.4)$$

because $E^*(k'')$ was set equal to $k''^2/2m$ for $k'' > 2.6k_F$. The last integral can be analytically evaluated;

$$\frac{1}{\pi} \int_0^\infty dk \frac{k^2 j_i(kr) j_i(kr')}{z - k^2/2m} = ma h_i(iar_>) j_i(iar_<), \quad (4.5)$$

where $a = [-2mz]^{1/2}$ (z is always < 0); j_i and h_i are l th-order spherical Bessel and H \ddot{a} nk \ddot{a} l functions, respectively, with $r_>$ designating the greater value of r and r' , and $r_<$ designating the lesser.

Wave functions:

$$\begin{aligned} \psi_{\nu\nu'} J^S(kr) &= j_i(kr) \delta_{\nu\nu'} + m \int_0^\infty dr' r'^2 G_\nu(r, r') \\ &\times \sum_{\nu''=J-1}^{J+1} V_{\nu\nu''} J^S(r', \bar{\nabla}) \psi_{\nu''\nu'} J^S(kr'), \end{aligned} \quad (4.6)$$

here written for both coupled and uncoupled ($\sum_{\nu''} \rightarrow \sum_{\nu''} \delta_{\nu''\nu'}$ and $\psi_{\nu\nu'} \rightarrow \delta_{\nu\nu'} \psi_i$) partial waves.

K matrices:

$$\begin{aligned} K_{kk} &= 4\pi \sum_{J_S} \sum_{l=J-1}^{J+1} C_{JlS} \int_0^\infty r^2 dr j_l(kr) \\ &\times \sum_{l'=J-1}^{J+1} V_{ll'} J^S(r, \bar{\nabla}) \psi_{ll'} J^S(r). \end{aligned} \quad (4.7)$$

C_{JlS} , the appropriate statistical weight, is given by

$$\frac{1}{3} [(2T+1)(2J+1)] (=1 \text{ for } S \text{ states}),$$

T being the isospin, and J being the total angular momentum of the state in question.

¹³ S. Coon and J. Dabrowski, Phys. Rev. **140**, B287 (1965).

Single-particle potential:

$$\begin{aligned} \mathcal{U}(k) &= \frac{6}{\pi^2} \int_0^{(k_F-k)/2} k'^2 dk' (k'|K|k') \\ &+ \frac{3}{\pi^2} \int_{|k_F-k|/2}^{(k_F+k)/2} k'^2 dk' (k'|K|k') \\ &\times \left(1 + \frac{k_F^2 - k^2 - 4k'^2}{4kk'} \right), \end{aligned} \quad (4.8a)$$

for $k < k_F$. For $k \geq k_F$, the first integral vanishes.

The self-consistent energy is

$$E(k) = k^2/2m + \mathcal{U}(k), \quad (4.8b)$$

and the binding energy per particle

$$= \int_0^{k_F} dk \left(\frac{k^2}{2m} + \frac{1}{2} \mathcal{U}(k) \right) / \int_0^{k_F} dk. \quad (4.9)$$

The arrows on top of $V_{\nu\nu'}$ and $V_{\nu\nu}$ in the wave-function and K -matrix equations mean ∇_r in $V(r, \bar{\nabla}_r)$ operates on G_ν and j_i , respectively, according to the index of ∇_r . This is consistent with the treatment of the momentum-dependent potential as used in Sec. III.

At this point mention should be made that when calculating off the energy shell (inelastic scattering), a new, sixth, invariant potential form (in coordinate space) can arise:

$$\sigma_1 \cdot (\mathbf{p}_f - \mathbf{p}_i) \sigma_2 \cdot (\mathbf{p}_e - \mathbf{p}_i) / m^2 \quad (4.10)$$

\mathbf{p}_i and \mathbf{p}_f being the initial and final scattering momentum operators, respectively. Since we wish to compare calculations made here with those of BM,¹² this sixth form will not be included. Moreover, since it is proportional to $\mathbf{p}_{i,f}^2/m^2$, its effects will not be noticed until high momenta, where the scattering process is assumed to be elastic and $\mathbf{p}_i - \mathbf{p}_f = 0$.

Again four-point Gaussian quadratures were used, this time in 10 regions:

$$r = (0-2), (2-10), (10-21), (21-41) \text{ (BeV}/\hbar c)^{-1}.$$

In the last region, $\psi_{\nu\nu}$ was set equal to $j_i \delta_{\nu\nu}$. The momentum mesh used was the same as in BM¹²:

$$k = (0.1, 0.3, 0.5, 0.7, 0.9, 1, 1.4, 1.8) k_F.$$

The Fermi momentum is given, in (BeV/ c), as $k_F = 1.524/r_0$, if r_0 is in (BeV/ $\hbar c$)⁻¹ or $k_F = 0.3006/r_0$, if r_0 is in fermis:

$$\left(\frac{4}{3} \pi r_0^3 \right)^{-1} = \rho = N/V = \frac{2 \times 2 \times \frac{4}{3} \pi k_F^3}{(2\pi\hbar)^3}.$$

The computation for each value of the Fermi momentum (including the exclusion-principle integral) took about 7.5 sec (30 points of r mesh), and one major iteration took about 12 min (2 min of which were used to tabulate Bessel functions, potentials, etc.). The calcu-

lations were performed on the CDC 3600 computer at the University of California, San Diego.

The equations actually solved (as in the scattering problem) were for the radial wave functions $u(kr) = kr\psi(kr)$ because of the resulting simplification of the functional form of the kernel of the integral equations.

V. CONCLUSIONS

A. Phase-Shift Analysis and the Deuteron

The experimental data for the phase shifts to be fit was obtained from analyses at 25-, 50-, 95-, 142-, 210-, and 310-MeV lab incident kinetic energy compiled by Noyes *et al.*¹⁴ A fit of the 3S_1 , 1S_0 , 3P_0 , 3P_1 , and 3P_2 was done at the six energies mentioned above.

$$\chi^2 = \sum_{i=\text{expt data}} \left\{ \frac{\delta_i(\text{calc}) - \delta_i(\text{expt})}{\Delta\delta_i(\text{expt error})} \right\}^2$$

= 446.5 (30 pieces of data)

was obtained for the following data: Taken as fixed, were the pion ($g_\pi^2=14$, $\mu_\pi=0.135$) (BeV/ c^2), the η ($g_\eta^2=14$, $\mu_\eta=0.548$) and the masses of the ω and φ ($\mu_\omega=0.78$, $\mu_\varphi=1.02$). These were picked for comparisons to SW¹.

The parameters were the σ ($g_\sigma^2=3.036$, $\mu_\sigma=0.461$), the ρ ($g_\rho^2=1.1$, $g_\rho^2=21.9$, $\mu_\rho=0.531$), the ω and φ couplings ($g_\omega^2=g_\varphi^2=3.03$), and the cutoffs [$\Lambda_1=2.32$, $\Delta\Lambda=-0.156$ is the increment of Λ_1 determining the other three cutoffs, $\Lambda_n=\Lambda_1+(n-1)\Delta\Lambda$].

This fit was achieved by varying each parameter until a local minimum in χ^2 was reached and then this over-all fit was checked by redoing the fit with each parameter.

A true χ^2 should reflect the correlation between errors contained in the associated "error matrix." However, the "uncorrelated χ^2 " calculated here is so large that it loses any real statistical meaning, and just serves as a rough comparative check between different potentials. Calculations employing fits to the phase shifts instead of directly fitting primary data (cross sections, polarizations, etc.) were done because of the ease of calculations and because it was felt the S states should be strongly weighted in the scattering analysis since this potential was going to be used for nuclear-matter calculations.

In order to understand the meaning of this fit—since other phase-shift analyses either use different data and/or rely on more primary experimental data (polarization and cross-section data)—it proves enlightening to do the same calculation with two other potentials widely in use, the Breit potential (81 parameters), more commonly referred to as the "Yale" potential,² and the Brueckner-Gammel-Thaler (BGT) potential, both re-

ported in BM.¹² The fits were done to the same data with no attempt to readjust parameters. Therefore, no quantitative conclusions should be made about the superiority of this potential over others; but one should recognize that "reasonable" fits were obtained with few parameters. Also, in Sec. V B, the phase shift will help to illustrate how this potential gives more binding in nuclear matter.

The BGT potential gave a χ^2 of 1596.6. The Breit potential, using a core of 0.506 F (isospin $T=0$ states should have a core of 0.5002 F, and isospin $T=1$ states should have a core of 0.5116 F), gave a χ^2 of 497. The infinite repulsive core was approximately simulated by a potential of 10 000 BeV in the core region.

However, the D states are rather difficult to fit, and without any further adjustment of parameters, a χ^2 was calculated for all S , P , and D waves (66 pieces of data): This potential (POT) gave $\chi^2=1483.5$, BGT gave $\chi^2=2540.4$, Breit (using the proper cores) gave $\chi^2=513.0$.

To solve the deuteron problem, the following procedure was used: The deuteron Schrödinger's equation can be recast into a set of coupled integral equations (with no inhomogeneous terms), and with the Green's function being the same function as the asymptotic Green's function for the nuclear-matter problem, Eq. (4.5) [the conjugate variable being $\gamma=(-m\epsilon)^{1/2}$, where ϵ is the binding energy of the deuteron]. The bound-state problem can then be written as

$$\begin{pmatrix} u \\ w \end{pmatrix}_{\gamma r} = m r \int d r' \begin{pmatrix} G_s & 0 \\ 0 & G_D \end{pmatrix}_{\gamma r, \gamma r'} \times \begin{pmatrix} V_S & V_T \\ V_T & V_D \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_{\gamma r' r'} \frac{1}{\epsilon} \quad (5.1a)$$

or as

$$K_1(\epsilon)u(\epsilon)=0, \quad w(\epsilon)=K_2(\epsilon)u(\epsilon), \quad (5.1b)$$

where u and w are the coupled S - and D -state wave functions [see Eq. (5.4)], respectively, and where K_1 and K_2 are matrices achieved by numerically approximating all integrals by finite sums. The condition for solubility of the ensuing set of linear equations represented by Eq. (5.1) is

$$\det |K_1(\epsilon)| = 0, \quad (5.2)$$

independent of the eigenfunctions u and w .

A "search" was done to find the value of ϵ satisfying the condition Eq. (5.2), thus giving the $N \times N$ matrix K_1 , N being the number of mesh points. The homogeneous problem, expressed as N linear equations, (5.1) is solved by using the determinantal condition, Eq. (5.2), to set one of the $u(r_i)$, $i=1, \dots, N$, equal to a constant (say 1). Then, $(N-1)$ entries of one column of the $N \times N$ matrix K_1 can be used as an inhomogeneous vector $V(K_1)$ for the set of $(N-1) \times (N-1)$ equations:

$$K_1' u' = -V(K_1), \quad (5.3)$$

¹⁴ M. H. MacGregor, R. A. Arndt, and A. A. Dubow, Phys. Rev. **135**, B628 (1964); M. H. MacGregor and R. A. Arndt, *ibid.* **139**, B362 (1965); H. P. Noyes, D. S. Bailey, R. A. Arndt, and M. H. MacGregor, *ibid.* **139**, B380 (1965).

which is solved for u' [a vector with $(N-1)$ entries] directly by matrix inversion, after which the eigenvector w can be calculated by Eq. (5.1).

At the correct energy eigenvalue, the deuteron wave function was used to calculate the electric quadrupole moment of the deuteron¹⁵ given as

$$Q = (\Phi, \frac{1}{2}r^2 P_2(\cos\theta)\Phi) / (\Phi, \Phi), \quad (5.4)$$

Φ being the total 3S_1 -state wave function—the sum of an S -state part and a D -state part, $\Phi = (u/r)P_0(\cos\theta) + (w/r)P_2(\cos\theta)$. Q is expressed as

$$Q = \left\{ \frac{1}{(50)^{1/2}} \int_0^\infty r^2 u w dr - \frac{1}{20} \int_0^\infty r^2 w^2 dr \right\} / \int_0^\infty (u^2 + w^2) dr. \quad (5.5)$$

At a binding energy of -8.09 MeV, a quadrupole moment of $0.212 F^2$ was calculated. (The experimental numbers are -2.22 MeV and $0.274 F^2$, respectively.)

It should be understood that the usual procedure in fitting the deuteron is to first put in the exact binding energy and then treat the central/tensor potential ratio as a parameter to be fit to an eigenvalue condition. Thus it is not clear just what the error in the binding energy is. In work currently being done on this problem, more care is being given to properly include the deuteron data (binding energy and quadrupole moment) as part of the χ^2 fit to define the interaction potential.

B. Nuclear Matter

Nuclear-matter calculations were done at $k_F = 0.22, 0.25, 0.275, 0.295$ BeV/ c , obtaining -14.6 -MeV binding at $k_F = 0.26$ ($\Rightarrow r_0 = 1.15 F$). (See Fig. 2.)

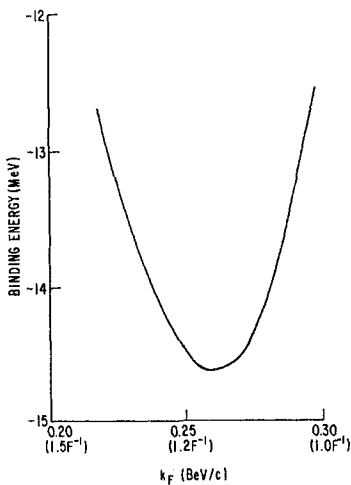


FIG. 2. Binding energy versus k_F for POT $\chi^2 = 446$. [$1 F = 5.07$ (BeV/ $\hbar c$)⁻¹.]

¹⁵ J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

One check on the calculations was to obtain equal phase shifts, for very low energies, between the nuclear matter (no exclusion principle and free energy spectrum) and the scattering computer codes. The scattering code was best checked by obtaining the same solutions by also solving the corresponding differential equation. The Gaussian-quadrature mesh gave at least three significant figure accuracy in the wave-function calculations.

As concerns the nuclear-matter calculation, probably the most important property of the nucleon-nucleon potential is the central/tensor ratio. This apparently non-unique ratio to be fit by the scattering data gives quite different answers for the binding energy, the larger ratio giving the greater binding.¹² On Fig. 3 is plotted the

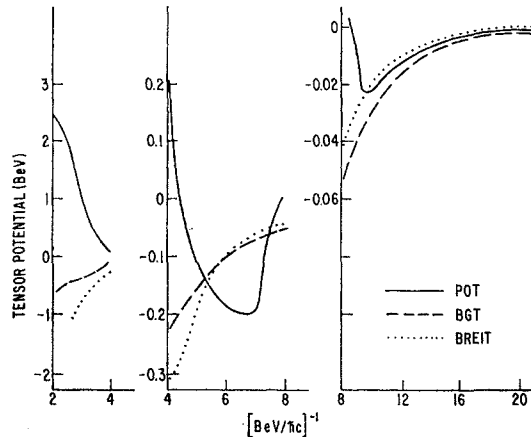


FIG. 3. Tensor potentials from POT, BGT, and Breit (POT is evaluated at 95-MeV lab energy). The "bump" at $r \sim 9$ is explained in the caption of Fig. 7.

diagonal part of the tensor force,

$$\frac{1}{mG(r,r)} \left(X + Y \frac{d}{dr} + Z \frac{d^2}{dr^2} \right) G(r,r), \quad (5.6)$$

versus the tensor forces of the BGT potential and the Breit potential. The large, short-range repulsion of the tensor force in the scattering problem was found to be unimportant as only slight changes in the scattering problem were produced upon erroneous inclusion of the hard core in the off-diagonal tensor potential when doing the BGT calculations. This is because the large repulsion of the central force drives the wave function $\rightarrow 0$ in this region. The much smaller attractive part of the POT tensor force is to be noted.

Although all three potentials give substantially the same nuclear-bar phase shifts for 3S_1 and 3D_1 , it is very elucidating to compare the ratio of

$$\tan^{-1}(j_0 V_0 \psi_1) / \tan^{-1}(j_0 V_T \psi_2) = R_{C/T}, \quad (5.7)$$

where V_0 and ψ_1 are the diagonal potential and wave function, respectively, and V_T and ψ_2 are the associated

coupled, off-diagonal potential and wave function (see Table I).

Although POT gives a more "reasonable" ratio for the 3S_1 matrix elements when compared to the 1S_0 phase shifts to which the 3S_1 nuclear-bar phase shifts are close, this larger central/tensor admixture can be better understood as a large central contribution of the σ meson which is not contained in the other potentials. The argument that this contribution is necessary, by considering either the σ meson or contributions from π - π exchanges, is well formulated in SW.¹

The qualitative differences due to this very large central/tensor ratio ($R_{C/T}$) explains why these calculations give more binding in nuclear matter. In Table II, the first column gives the ratio of the two 3S_1 matrix elements (the sum of which contributes to the K matrix). The denominator is the matrix element of the tensor potential. The second column gives the contribution to the K matrix when the tensor potential is set identically to zero. The third column gives the fraction

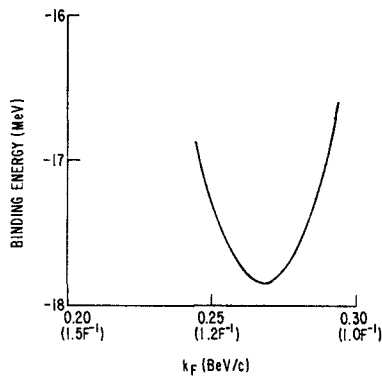


FIG. 4. Binding energy versus k_F for POT $\chi^2=770$.

of attraction gained with $V_T=0$ [i.e., (value with $V_T=0$ - full value)/(full value)].

To measure the sensitivity of the binding energy and equilibrium density to these potentials, a calculation was done using parameters giving a $\chi^2=770$ for the 3S_1 , 1S_0 , ${}^3P_{0,1,2}$ states: $g_{\pi^2}=g_{\eta^2}=14$, $g_{\sigma^2}=3.05$, $g_{\rho_1^2}=1.1$, $g_{\rho_2^2}=21.9$, $g_{\omega^2}=g_{\varphi^2}=3$, $\mu_{\pi}=0.135$ (BeV/c²), $\mu_{\sigma}=0.45$, $\mu_{\rho}=0.531$, $\mu_{\omega}=0.78$, $\mu_{\varphi}=1.02$, $\mu_{\eta}=0.548$, $\Delta\Lambda=-0.152$, $\Lambda_1=2.32$. These numbers are, not very different from those giving $\chi^2=446$, but a greater $R_{C/T}$ ratio did exist for $\chi^2=770$, showing some sensitivity in even the scattering problem calculation to the effective central-to-tensor ratios. The result of the nuclear-matter calculation was 17.8-MeV binding at a density of $k_F=0.27$ ($r_0=1.11$ F) for this potential (see Fig. 4).

As the D waves were not carefully fitted to the scattering data, a comparison of D -wave contributions was made to that of the Breit potential which does fit the scattering data rather well. The Breit potential gave a D -state contribution of -7.27 MeV to the binding energy, while POT gave a D -state contribution of -8.73

MeV, a difference of 1.46 MeV. Inclusion of the F , G , and H partial waves, obtained directly from the experimental phase shifts gave a repulsion of 0.2. If one used just Breit's D states, a binding energy of -12.9 is obtained, still 4.6 MeV lower than the Breit potential gave for the entire calculation. This result is explained by the higher central/tensor ratio, as discussed above. [It should be stressed that phenomenological potentials which differ—in the sense of different forms of Eq. (2.10)—in different angular momentum states have an ambiguous "meaning" when applied to different physical problems, as the nucleons are physically described by the total wave function.]

Lab energy (BeV)	POT	Breit	BGT
0.025	1.44 / 0.254	-1.13 / 1.435	1.027 / 1.434
0.095	1.033/-0.576	-0.745 / 1.096	0.031 / 0.78
0.21	0.754/-0.664	-0.7366/0.919	-0.197 / 0.503
0.31	0.627/-0.671	-0.761 / 0.835	-0.3161/0.394

Calculations at $k_F=0.275$ of the $\chi^2=770$ potential, using the resultant self-consistent spectrums were done to test the validity of certain other approximations:

(1) The reference spectrum approximation,¹⁶ dropping

$$\int_0^{2.6k_F} dk' \left\{ \frac{f(P,k')}{E(k)-E^*(k')} \frac{1}{E(k)-k'^2/2m} \right\} \times j_i(k'r) j_i(k'r') \quad (5.8)$$

(done for both P and D waves), gave the same K matrix as calculations done by including the above integral as part of the Green's function to within 0.1%. This is seen to be due to cancellation of

$$\int_0^{k_F} dk' \frac{(-1)}{E(k)-k'^2/2m} j_i j_i, \quad (5.9)$$

when $f(P,k')$ is $\equiv 0$, with the rest of the integral where $f(P,k') \sim 1$.

TABLE II. $R_{C/T}$ for K matrix [$(\hbar c/\text{BeV})^2/\hbar c$].

k/k_F	POT			Breit		
	Full	$V_T=0$	%	Full	$V_T=0$	%
0.1	-260.0/112.8	-255.6	0.625	5.6/-103.5	-11.7	-0.882
0.5	-176.3/98.7	-167.9	1.16	43.2/-89.6	25.7	-1.92
0.9	-91.3/67.3	-84.5	4.63	56.1/-66.5	36.9	-4.55

¹⁶ H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. 129, 225 (1963).

(2) Use of the free-particle energy spectrum for states above the Fermi surface instead of the self-consistent one in the S -state Green's function gave a binding energy of -15.47 MeV, or 1.5 MeV less binding, than the calculation using the self-consistent particle energies at $k_F=0.275$.

(3) To illustrate the importance of recognizing the momentum-dependent nature of POT, a binding-energy calculation was done using the self-consistent spectrum of POT, $\chi^2=770$, letting $d/dr \rightarrow k$ (the "elastic" scattering potential) instead of $d/dr \rightarrow [2mE(k)]^{1/2}$ when operating on the asymptotic Green's functions. The result was -39 -MeV, instead of -17 -MeV, binding due to the Green's functions' not "digging" enough into the shorter-range repulsive region of the potential because $[2ME(k)]^{1/2} > k$. These potentials, properly treated, are also nonlocal; that is, the potential is not a function of r , but rather of r and r' when used as

$$\psi(r) = \chi(r) + m \int dr' V(r, r') G(r', r) \psi(r'),$$

$$mV(r, r') = \frac{1}{G(r, r')} \left(X(r') + Y(r') \frac{d}{dr'} + Z(r') \frac{d^2}{dr'^2} \right) G(r, r'). \quad (5.10)$$

The nonlocality can usually be completely shouldered onto the d/dr' term as d^2/dr'^2 can be reexpressed as $(L^2/r'^2 \pm \alpha^2)$ operating on simple Green's functions by use of the homogeneous Bessel's equation. α corresponds to the proper conjugate variable to the radial parameter in the Green's function, and the $+$ or $-$ sign is to be taken depending on whether a bound-state or scattering problem, respectively, is to be done. The d/dr' term cannot be so treated:

The local part of the potential for the 1S_0 state is

$$X(r') - k^2 Z(r') \quad (5.11)$$

and to this is added the nonlocal part (for the scattering calculation)

$$Y(r') k \frac{d}{d(kr')} [\sin(kr_<) \cos(kr_<)] / [\sin(kr_<) \cos(kr_>)], \quad (5.12)$$

where $r_<$ is the lesser of r and r' , and $r_>$ is the greater. On Fig. 5 is plotted the local potential above and $Y(r')$ for $k^2=0.0445$.

(In the nuclear-matter calculation, there is additional nonlocality from the d^2/dr'^2 term operating inside the integral on that part of the Green's function containing the exclusion-principle correction.)

Exhibited on Fig. 6 is the diagonal 3S_1 potential in nuclear matter at a value of $k/k_F=0.1$ ($k_F=0.275$ BeV/ c). On Fig. 7 is plotted similar data for the off-diagonal tensor

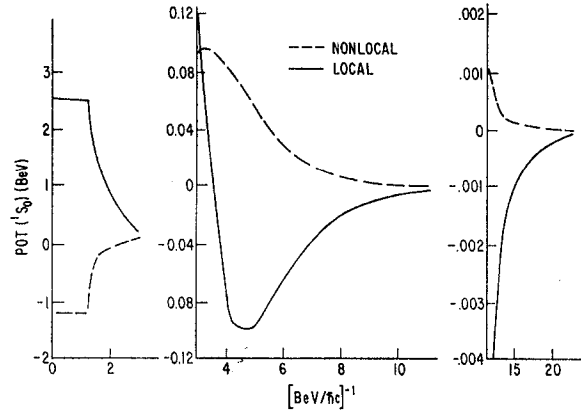


FIG. 5. Local and nonlocal contributions to the 1S_0 -state potential, $k^2=0.0445$ (corresponds to 95-MeV incident lab energy).

force. The diagonal part includes local and nonlocal contributions. The 3S_1 was calculated with the exclusion-principle correction.

That the S states are so sensitive to the shape of this potential in the small- r regions is, of course, no less physical than the sensitivity of similar calculations to the "hard-core" radius, as was demonstrated when using the Breit potential.

As can be seen from Eq. (5.10), the contribution of the diagonal nonlocal potential to the local potential is

$$Y(r) k \cot(2kr), \quad (5.13)$$

($= Y(r) \alpha \coth(2\alpha r)$ for the bound-state calculation).

This is obtained from contributions at points r' approaching r from above and below:

$$(kr) \psi(kr) = (kr) \chi(kr) + (m/k) \cos(kr) \times \int_0^r \sin(kr') V(r', \nabla_{r'}) \psi(kr') r' dr' + (m/k) \sin(kr) \times \int_r^\infty \cos(kr') V(r', \nabla_{r'}) \psi(kr') r' dr'. \quad (5.14)$$

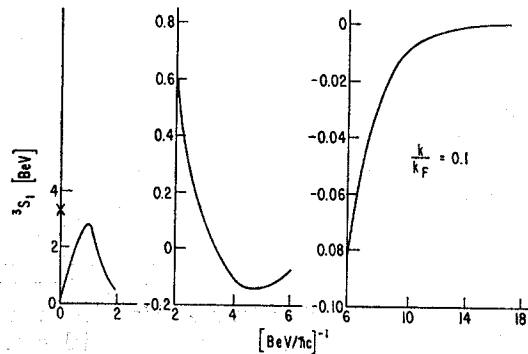


FIG. 6. 3S_1 POT for $k/k_F=0.1$, $k_F=0.275$ BeV/ c [$1 F=5.07$ (BeV/ $\hbar c$) $^{-1}$].

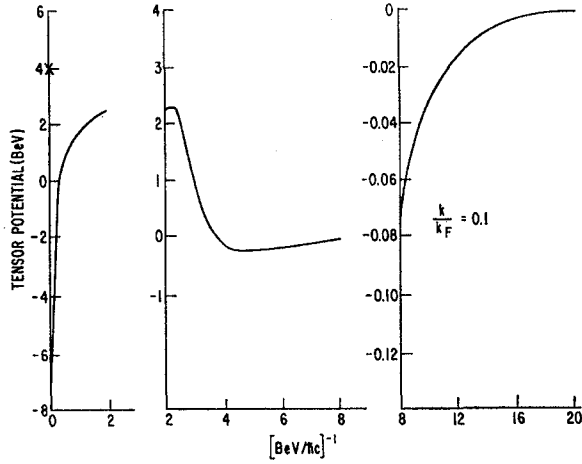


FIG. 7. Tensor POT for $k/k_F=0.1$, $k_F=0.275$ BeV/c [$1F=5.07$ (BeV/hc) $^{-1}$]. Comparison between this figure and Fig. 3 illustrates the important, different contributions between the scattering and bound-state problems, from the nonlocal potential. [Notice that $kr=\pi\approx(0.0445)^{1/2}\times 9$ "explains" the anomalous bump in Fig. 3. At this value of kr , however, the contribution of the product of the potential and the Green's function is zero.]

However, as $r \rightarrow 0$, the first integral in the right-hand side of Eq. (5.14) vanishes and in contrast to Eq. (5.13), one obtains

$$-Y(r)k \tan(kr),$$

which, for small kr

$$= -Y(r)k^2r, \quad (5.15)$$

($=Y(r)\alpha$ for the bound-state calculation) and does not diverge for small r . This is necessary to satisfy the S -state boundary conditions. Unlike the " δ -function" behavior of the d^2/dr^2 term, this "step-function" behavior of the d/dr term cannot be explicitly taken out from the integral, and one must include this somewhat discontinuous behavior in order to properly carry out the mathematics. As can be seen from Fig. 5, the $Y(r)$ term quickly becomes insignificant when compared to the local part of the potential, as r increases, so that these sensitive, nonlocal effects are only important for the S states. (Of course, integration by parts would express the nonlocality in terms of the logarithmic derivative of the wave function. This might be desirable for some other physics calculations.)

If one treats the momentum dependence correctly, there might exist an approximation for obtaining a local energy-dependent potential by approximating $d/dr \rightarrow \alpha$. This is seen to be true for both the scattering and the nuclear-matter calculations. Only the S states (especially the 3S_1 state) in each case, really suffer from this approximation. Of course, the larger α is, the worse the approximation. Notice should be taken, however, that because of such error in the S states the over-all calculations are qualitatively quite poor, illustrating the importance of nonlocality in the "core" (small r) region. (See Tables III and IV.) It may be observed that if both

TABLE III. Nonlocality in nucleon scattering.

	Lab. energy (BeV)	Phase shifts	
		With G' , $\chi^2=1438.5$	With $\alpha G(\alpha=k)$, $\chi^2=93\ 560$
1S_0	0.025	0.895	0.973
	0.142	0.249	0.335
	0.31	-0.192	-0.06
3P_1	0.025	-0.093	-0.093
	0.142	-0.268	-0.276
	0.31	-0.321	-0.331
3D_2	0.025	0.064	0.065
	0.142	0.566	0.552
	0.31	0.712	0.654
3S_1	0.025	1.48 / -0.432	-1.56/1.45
	0.142	0.888 / -0.572	0.718 / -0.192
$(R_{C/T})$	0.31	0.625 / -0.668	0.448 / -0.363

TABLE IV. Nonlocality in nuclear matter ($k_F=0.275$ BeV/c).

	K matrix [(hc/BeV) 2 /hc]		
	k/k_F	With G'	With $\alpha G[\alpha=(2mE(k))^{1/2}]$
1S_0	0.3	-99.5	-95.9
	0.7	-38.9	-36.8
3P_1	0.3	10.75	10.78
	0.7	15.64	15.74
3D_2	0.3	-4.26	-4.25
	0.7	-17.8	-17.5
3S_1	0.3	-222.8/107.3	-244.3/175.9
	$(R_{C/T})$	0.7	-130.9/88.2
Binding energy		-14.48 MeV	+8.48

momentum dependence and nonlocal effects are treated "incorrectly," the two deviations from the correct procedure somewhat cancel each other. This has given hope that a *more* phenomenological set of static potentials might be derived, which would be more tractable for other nuclear-physics calculations. Such calculations are now in progress.

In summation, one concludes that this one-meson-exchange potential has consistently, in most cases quantitatively, and in all cases qualitatively explained many nonrelativistic observables of nuclear physics—with only eight parameters, which themselves are reasonable physical quantities.

ACKNOWLEDGMENTS

The author thanks Professor Keith A. Brueckner for offering continual encouragement throughout the long span of time these calculations encompassed; the author enjoyed many fruitful conversations with Professor Brueckner associated with details and interpretation of the nuclear-matter and scattering calculations. Professor David Y. Wong also offered encouragement and insight into the scattering problem in the early stages of this work and again gave insight into the interpretation of results associated with the meson interactions as defined here. The hospitality of the Niels Bohr Institute

from June through October, 1964, as offered by Professor Aage Bohr and Professor Gerry Brown was greatly appreciated. In particular, it was through discussions in Copenhagen with Dr. Tony Green, Dr. David Brandow, and Dr. C. Wong that much detailed knowledge of many-body theory was gained. Mrs. Marielle Bryant was of immeasurable help in organizing many cumbersome programs, and further, gave invaluable aid and suggestions in the running of some of these programs while the author was a consultant to RAND Corporation from April, 1965 through April, 1966. The UCSD CDC 3600 computer staff made the tasks of computing simple and efficient enough so as to place the responsibility of the computations entirely upon the author. The author also thanks Toby Burnett for pointing out an error in the original derivation of the $\sigma \cdot \mathbf{p}$ potential. The calculation in Appendix B was derived by the author after being informed of this error.

APPENDIX A: POTENTIAL (POT)

For each meson considered,⁷ five F functions (F_c , F_σ , F_T , F_{LS} , $F_{\sigma p}$) will be given. For the first four, the $1/E$ phase-space factor is to be later expanded to $(1/m)(1 - p^2/2m^2)$. The fifth, $F_{\sigma p}$, is already multiplied by $(1/m^2)\sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p}$ and proportional to p^2/m^2 , so $1/E$ is replaced by $1/m$. It is understood the F 's are to be regularized by the methods given in Sec. II, whereupon they are to be used as the M 's in Eq. (2.10'), after the nonrelativistic reduction, to obtain $V(r, p^2)$.

Integrals over momentum transfer space for F_c , F_σ , $F_{\sigma p}$ give, with

$$F = +g_\mu^2 m^2 / E(\mu^2 + \Delta^2), \quad (\text{A1})$$

a term $-g_\mu^2 e^{-\mu r} / r$.

Integrals of $F_{LS} = +g_\mu^2 / E(\mu^2 + \Delta^2)$ give a term

$$-\frac{g_\mu^2 e^{-\mu r}}{r} \left(\frac{1}{\mu^2 r^2} + \frac{1}{\mu r} \right) \frac{\mu^2}{m^2}. \quad (\text{A2})$$

Integrals of $F_T = g_\mu^2 / E(\mu^2 + \Delta^2)$ give a term

$$-\frac{g_\mu^2 e^{-\mu r}}{r} \left(\frac{3}{\mu^2 r^2} + \frac{3}{\mu r} + 1 \right) \frac{\mu^2}{m^2}. \quad (\text{A3})$$

The F 's given are the $I=0$ (isospin=0) potentials due to the exchange of $I=0$ particles. The other three possibilities are obtained from the isospin crossing matrix:

$$\begin{aligned} (\text{isospin of potential}) &= (\text{crossing matrix}) \\ &\quad \times (\text{isospin of particle}), \end{aligned}$$

or

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{A4})$$

Pseudoscalar meson (π, η):

$$\begin{aligned} F_c &= 0, \\ F_\sigma &= \frac{g^2}{12E} \frac{\Delta^2}{\mu^2 + \Delta^2}, \\ F_T &= -\frac{g^2}{12E} \frac{1}{\mu^2 + \Delta^2}, \end{aligned} \quad (\text{A5})$$

$$F_{LS} = 0,$$

$$F_{\sigma p} = 0.$$

Scalar meson (σ):

$$\begin{aligned} F_c &= \frac{g^2}{E(\mu^2 + \Delta^2)} \left(m^2 + \frac{\Delta^2}{4} + \frac{\Delta^4}{64m^2} - \frac{\Delta^2 p^2}{16m^2} - \frac{\Delta^4 p^2}{128m^4} \right), \\ F_\sigma &= \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^4}{64m^2} - \frac{\Delta^2 p^2}{24m^2} - \frac{\Delta^4 p^2}{128m^4} \right), \\ F_T &= \frac{g^2}{E(\mu^2 + \Delta^2)} \left(-\frac{p^2}{48m^2} \right), \end{aligned} \quad (\text{A6})$$

$$F_{LS} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{2} + \frac{\Delta^2}{16m^2} - \frac{p^2}{8m^2} - \frac{\Delta^2 p^2}{32m^4} \right),$$

$$F_{\sigma p} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{16} \right).$$

Vector meson (ρ, ω, φ):

(1) Vector coupling:

$$\begin{aligned} F_c &= \frac{g^2}{E(\mu^2 + \Delta^2)} \left(-m^2 + \frac{\Delta^2}{2} - \frac{\Delta^4}{64m^2} \right. \\ &\quad \left. - 2p^2 + \frac{\Delta^2 p^2}{16m^2} + \frac{\Delta^4 p^2}{128m^4} \right), \\ F_\sigma &= \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{6} - \frac{\Delta^4}{64m^2} + \frac{\Delta^2 p^2}{24m^2} + \frac{\Delta^4 p^2}{128m^4} \right), \end{aligned} \quad (\text{A7})$$

$$F_T = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{12} + \frac{p^2}{48m^2} \right),$$

$$F_{LS} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(\frac{3}{2} - \frac{\Delta^2}{16m^2} + \frac{p^2}{8m^2} + \frac{\Delta^2 p^2}{32m^4} \right),$$

$$F_{\sigma p} = \frac{g^2}{E(\mu^2 + \Delta^2)} \left(-\frac{\Delta^2}{16} \right);$$

(2) Tensor coupling (for the ρ only):

$$\begin{aligned}
 F_c &= \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(-\frac{\Delta^4}{16m^2} + \frac{\Delta^6}{256m^4} - \frac{\Delta^4 p^2}{64m^4} - \frac{\Delta^6 p^2}{512m^6} \right), \\
 F_\sigma &= \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{6} - \frac{\Delta^4}{8m^2} + \frac{\Delta^6}{256m^4} \right. \\
 &\quad \left. + \frac{\Delta^2 p^2}{3m^2} - \frac{\Delta^4 p^2}{96m^4} - \frac{\Delta^6 p^2}{512m^6} \right), \\
 F_T &= \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{12} + \frac{p^2}{6m^2} - \frac{\Delta^2 p^2}{192m^4} \right), \\
 F_{LS} &= \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(-\frac{3\Delta^2}{8m^2} + \frac{\Delta^4}{64m^4} - \frac{\Delta^2 p^2}{32m^4} - \frac{\Delta^4 p^2}{128m^6} \right), \\
 F_{\sigma p} &= \frac{g_2^2}{E(\mu^2 + \Delta^2)} \left(-\frac{\Delta^2}{2} + \frac{\Delta^4}{64m^2} \right);
 \end{aligned} \tag{A8}$$

(3) Mixed vector-tensor coupling:

$$\begin{aligned}
 F_c &= \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{2} - \frac{\Delta^4}{8m^2} + \frac{\Delta^2 p^2}{2m^2} + \frac{\Delta^4 p^2}{32m^4} \right), \\
 F_\sigma &= \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(\frac{\Delta^2}{3} - \frac{\Delta^4}{8m^2} + \frac{\Delta^2 p^2}{3m^2} + \frac{\Delta^4 p^2}{32m^4} \right), \\
 F_T &= \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(\frac{1}{6} + \frac{p^2}{6m^2} \right), \\
 F_{LS} &= \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(2 - \frac{\Delta^2}{2m^2} + \frac{p^2}{m^2} + \frac{\Delta^2 p^2}{8m^4} \right), \\
 F_{\sigma p} &= \frac{gg_2}{E(\mu^2 + \Delta^2)} \left(-\frac{\Delta^2}{2} \right).
 \end{aligned} \tag{A9}$$

APPENDIX B: $\sigma \cdot \mathbf{p}$ POTENTIAL

To express $\frac{1}{2}[\sigma_1 \cdot \mathbf{p} v(r) \sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \mathbf{p} v(r) \sigma_1 \cdot \mathbf{p}]$ as an eigenoperator of \mathbf{J}^2 , \mathbf{L}^2 , \mathbf{S}^2 write

$$\mathbf{p} = \nabla / i, \tag{B1}$$

$$\begin{aligned}
 \nabla &= \hat{r}(\hat{r} \cdot \nabla) - \hat{r} \times (\mathbf{r} \times \nabla) / r \\
 &= \hat{r} \partial / \partial r - i \hat{r} \times \mathbf{L} / r,
 \end{aligned} \tag{B2}$$

and

$$\sigma \cdot \nabla = \sigma \cdot \hat{r}(\hat{r} \cdot \nabla) - i \sigma \cdot \mathbf{r} \times \mathbf{L} / r^2. \tag{B3}$$

Using the Dirac identity,

$$\sigma \cdot \mathbf{r} \sigma \cdot \mathbf{L} = \mathbf{r} \cdot \mathbf{L} + i \sigma \cdot \mathbf{r} \times \mathbf{L} = i \sigma \cdot \mathbf{r} \times \mathbf{L}, \tag{B4}$$

Eq. (B3) may be written as

$$\sigma \cdot \nabla = \sigma \cdot \hat{r}[(\hat{r} \cdot \nabla) - \sigma \cdot \mathbf{L} / r]. \tag{B5}$$

Then,

$$\sigma_1 \cdot \nabla v(r) \sigma_2 \cdot \nabla = \sigma_1 \cdot \hat{r}[\hat{r} \cdot \nabla - \sigma_1 \cdot \mathbf{L} / r] \times v \sigma_2 \cdot \hat{r}[\hat{r} \cdot \nabla - \sigma_2 \cdot \mathbf{L} / r] \tag{B6}$$

and

$$\begin{aligned}
 \sigma_1 \cdot \nabla v(r) \sigma_2 \cdot \nabla &= v(r) \sigma_1 \cdot \hat{r}[\hat{r} \cdot \nabla - \sigma_1 \cdot \mathbf{L} / r] \\
 &\quad \times \sigma_2 \cdot \hat{r}[\hat{r} \cdot \nabla - \sigma_2 \cdot \mathbf{L} / r] + [dv(r)/dr] \\
 &\quad \times \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r}[\hat{r} \cdot \nabla - \sigma_2 \cdot \mathbf{L} / r].
 \end{aligned} \tag{B7}$$

The first term on the right-hand side of Eq. (B7) is $v(r) \sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p}$, and will be further reduced. Expanding

$$\begin{aligned}
 \sigma_1 \cdot \nabla \sigma_2 \cdot \nabla &= \sigma_1 \cdot \hat{r}(\hat{r} \cdot \nabla) \sigma_2 \cdot \hat{r}(\hat{r} \cdot \nabla) \\
 &\quad + \sigma_1 \cdot \hat{r}(\sigma_1 \cdot \mathbf{L} / r) \sigma_2 \cdot \hat{r}(\sigma_2 \cdot \mathbf{L} / r) - \sigma_1 \cdot \hat{r}(\hat{r} \cdot \nabla) \sigma_2 \cdot \hat{r} \\
 &\quad \times (\sigma_2 \cdot \mathbf{L} / r) - \sigma_1 \cdot \hat{r}(\sigma_1 \cdot \mathbf{L} / r) \sigma_2 \cdot \hat{r}(\hat{r} \cdot \nabla);
 \end{aligned} \tag{B8}$$

using

$$\nabla_i r_j = r_j \nabla_i + \delta_{ij}, \quad i, j = 1, 2, 3$$

(cartesian coordinates)

$$\frac{1}{r} = \frac{1}{r} - \nabla_i \frac{r_i}{r^3}, \tag{B9}$$

$$\nabla_i \hat{r}_j = \nabla_i \frac{r_j}{r} = \frac{r_j}{r} \nabla_i + \frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3},$$

and

$$\sigma \cdot \mathbf{L} = \mathbf{r} \cdot \nabla - \sigma \cdot \mathbf{r}(\sigma \cdot \nabla), \tag{B10}$$

the following commutation relations can be easily derived:

$$\begin{aligned}
 [\hat{r} \cdot \nabla, \sigma_2 \cdot \hat{r}] &= 0, \\
 [\sigma_1 \cdot \mathbf{L}, \sigma_2 \cdot \mathbf{r}] &= \sigma_2 \cdot \mathbf{r} - \sigma_1 \cdot \mathbf{r} \sigma_1 \cdot \sigma_2, \\
 [\sigma_2 \cdot \mathbf{L}, 1/r] &= 0, \\
 [\sigma_2 \cdot \mathbf{L}, \mathbf{r} \cdot \nabla] &= 0.
 \end{aligned} \tag{B11}$$

Defining $2\mathbf{S} = \sigma_1 + \sigma_2$ and $S_{12} = 3\sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2$, and using

$$\sigma_1 \cdot \mathbf{L} \sigma_2 \cdot \mathbf{L} = 2(\mathbf{S} \cdot \mathbf{L})^2 - \mathbf{L}^2 + \mathbf{S} \cdot \mathbf{L}, \tag{B12}$$

then $\frac{1}{2}[\sigma_1 \cdot \mathbf{p} \sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \mathbf{p} \sigma_1 \cdot \mathbf{p}]$ can be written as

$$\begin{aligned}
 &\frac{1}{3}(S_{12} + \sigma_1 \cdot \sigma_2) \mathbf{p}^2 + [S_{12} + \frac{2}{3}(S_{12} + \sigma_1 \cdot \sigma_2) \mathbf{S} \cdot \mathbf{L}] \\
 &\quad \times \left(\frac{1}{r} \frac{d}{dr} - \frac{\mathbf{S} \cdot \mathbf{L}}{r^2} \right).
 \end{aligned} \tag{B13}$$

Finally,

$$\begin{aligned}
 &\frac{1}{2}[\sigma_1 \cdot \mathbf{p} v(r) \sigma_2 \cdot \mathbf{p} + \sigma_2 \cdot \mathbf{p} v(r) \sigma_1 \cdot \mathbf{p}] \\
 &= v(r) \left\{ \frac{1}{3}(S_{12} + \sigma_1 \cdot \sigma_2) \mathbf{p}^2 + [S_{12} + \frac{2}{3}(S_{12} + \sigma_1 \cdot \sigma_2) \mathbf{S} \cdot \mathbf{L}] \right. \\
 &\quad \left. \times \left(\frac{1}{r} \frac{d}{dr} - \frac{\mathbf{S} \cdot \mathbf{L}}{r^2} \right) \right\} \\
 &\quad + \frac{dv(r)}{dr} \frac{1}{3}(S_{12} + \sigma_1 \cdot \sigma_2) \left(\frac{d}{dr} - \frac{\mathbf{S} \cdot \mathbf{L}}{r} \right).
 \end{aligned} \tag{B14}$$