MATHEMATICAL COMPARISON OF COMBAT COMPUTER MODELS TO EXERCISE DATA

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ABSTRACT

The powerful techniques of modern nonlinear statistical mechanics are used to compare battalion-scale combat computer models (including simulations and wargames) to exercise data. This is necessary if large-scale combat computer models are to be extrapolated with confidence to develop battle-management, C^3 and procurement decision-aids, and to improve training. This modeling approach to battalion-level missions is amenable to reasonable algebraic and/or heuristic approximations to drive higher-echelon computer models.

Each data set is fit to several candidate short-time probability distributions, using methods of ‘‘very fast simulated re-annealing’’ with a Lagrangian (time-dependent algebraic cost-function) derived from nonlinear stochastic rate equations. These candidate mathematical models are further tested by using path-integral numerical techniques we have developed to calculate long-time probability distributions spanning the combat scenario.

We have demonstrated proofs of principle, that battalion-level combat exercises can be well represented by the computer simulation JANUS(T), and that modern methods of nonlinear nonequilibrium statistical mechanics can well model these systems. Since only relatively simple drifts and diffusions were required, in larger systems, e.g., at brigade and division levels, it might be possible to ‘‘absorb’’ other important variables (C^3, human factors, logistics, etc.) into more nonlinear mathematical forms. Otherwise, this battalion-level model should be supplemented with a ‘‘tree’’ of branches corresponding to estimated values of these variables.

I. INTRODUCTION: C^2 IN TRAINING AND COMPUTER MODELS

A. NECESSITY OF COMPARING COMPUTER MODELS TO EXERCISE DATA

This project addresses the utility of establishing an approach to compare exercise data to large-scale computer models whose relatively underlying microscopic interactions among men and machines are driven by the natural laws of Physics.

In this paper, the term computer model will be used to include computer simulations as well as computer wargames, the latter involving human participants in real time. I.e. it appears that JANUS(L) simulation can be compared favorably to JANUS(L) wargame. In this study, the focus will be to compare JANUS(T) wargame to National Training Center (NTC) data, since both systems then take into account human interactions.

It also should be noted that ‘‘large-scale’’ here refers to battalion-level. (Army systems scale by factors of 3-5, from company to battalion to brigade to division to corps to army.) If these battalion-level computer models can be favorably compared, and if consistency can be achieved between the hierarchy of large-scale battalion-level, larger-scale corps-level, and largest-scale theater-level computer models, then these higher echelon computer models also can be favorably compared. This could only enhance the value of training on these higher echelon computer models [1].

The necessity of depending more and more on combat computer models (including simulations and wargames) has been brought into sharper focus because of many circumstances, e.g.: (a) the nonexistence of ample data from previous wars and training operations, (b) the rapidly shortening time-scale on which tactical decisions must be made, (c) the rapidly increasing scale at which men and machines are to be deployed, (d) the increasing awareness of new scenarios which are fundamentally different from historical experiences, (e) and the rapidly increasing expense of conducting training exercises.

Furthermore, such computer models could be used to augment training. We presently spend several million dollars to cycle each battalion through NTC. The training of these commanders could be greatly enhanced if relatively inexpensive pre- and post-training wargames were provided which statistically replicate their training missions. Even, or rather especially, for the development of such training aids, proper analysis and modeling is required to quantitatively demonstrate that the computer models are good statistical representations of the training mission.

However, the level of acceptance of computer models in major military battle-management and procurement decisions appears to be similar to the level of acceptance of computer simulations in physics in the 1960’s. In physics, prior to the 1960’s, theory and experiment formed a close bond to serve to understand nature. In the 1960’s, academicians were fascinated with evolving computer technology, but very few people seriously accepted results from computer simulations as being on a par with good theory and good experiment. Now, of course, the situation is quite different. The necessity of understanding truly complex systems has placed computer simulation, together with theory and experiment, as an equal leg of a tripod of techniques used to investigate physical nature.

The requirements necessary to bring combat computer models to their needed level of importance are fairly obvious. In order to have confidence in computer-model data, responsible decision-makers must be convinced that computer models model reality, not metaphors of reality, models of models, or models of models of models, etc. Many people feel that not much progress has been made in the last decade [2,3] with regard to this issue, despite a general awareness of the problem.

If a reasonable confidence level in computer models of large-scale combat scenarios could be obtained, there are several immediate payoffs to be gained. More objective data could be presented for procurement decisions, e.g., provided by sensitivity analyses of sets of computer models differing in specific weapons characteristics. In order to give proper weight to these differing characteristics, their influence within the global context of full combat scenarios would be tested.

B. LARGE-SCALE C^2 AND NEED FOR MATHEMATICAL MODELING

Modeling phenomena is as much a cornerstone of 20th century Science as is collection of empirical data [4]. In essentially all fields of Science, mathematical models of the real world become tested by fitting some parameters to empirical data. Since the real world is often nonlinear and stochastic, it is not surprising that often this fitting process must involve fitting statistical, nonlinear, functional forms to data.

As in other fields of Science, in the context of modeling combat, reductionist doctrine is simply inadequate to fully understand large-scale systems. For example, a threshold is quickly reached at a level of any large system, be it physical, biological or social, when a ‘‘language’’ shift is required for effective command and control. A high-level commander cannot use a grease-board to track individual units, albeit he might periodically sample his units, but he must rather look at the overall systematics, e.g., aggregated measures of force (MOF) or effectiveness (MOE), attrition, resupply, etc. At this level we properly require command and control (C^2), rather than ‘‘supra-battle-management’’ from commanders. At this level we denote the system as large-scale. (See Fig. 1.)

Figure 1

This issue of utilizing MOF’s and MOE’s, e.g., starting at about battalion-level of combat, is relevant to computer models as well as to actual combat. Merely aggregating data to form MOF’s or MOE’s does not determine if results from one mission (combat or computer-model scenario) are comparable to another mission. E.g., small differences in tempo or in spatial distribution of FLOT (forward line of own troops), or FEBA (forward edge of the battle area), may cause tables of numbers to appear quite different.

Mathematical models of aggregated data should be expected to uncover ‘‘mechanisms’’ of combat, e.g., like line-firing or area-firing in simple Lanchester theory. More complex missions plausibly will contain more subtle mechanisms as well as weighted contributions of more basic mechanisms. Using this as hindsight, in some systems it may then be possible to specify a figure of merit, some simple set of numbers to capsulate the influence of these mechanisms.

These mechanisms are to be articulated by particular algebraic forms. Indeed, this is the most important sense of Physics, to use mathematical forms to articulate mechanisms and models of empirical phenomena. For example, it is to be expected that the use of this statistical mechanical approach will facilitate the process of identifying algebraic forms relevant to combat, because in many cases these algebraic forms will be sufficiently similar to other well-known physical mechanics determined in other physical studies.

Awareness of such plausible mechanisms permits the analyst or the field commander to more easily uncover variations of patterns or common themes unfolding in databases or in real time. In complex missions, such tools are more than niceties; they can be necessities.

Therefore, to be most useful, computer-model data should be aggregated and then mathematically modeled using variables as close as possible to the level of command to which they are to become decision-aids. The mathematical modeling must by necessity be nonlinear, e.g., offering alternative choices of possible outcomes, upon which human judgement and experience can be brought to bear and be accountable. The mathematical modeling must by necessity also be probabilistic, so that expected gains can be humanly weighed with respect to both payoffs and the probabilities of payoffs of alternative dynamic (time-dependent) states of the system.

It must be emphasized that this approach requires an evolution of knowledge. This project is developing models suitable to describe the statistical nature of selected force-on-force battalion and brigade scenarios. It is expected that the accumulation of models of many types of scenarios will lead to a better fundamental understanding of combat with direct operational applications.

As is often discussed [5], too often we have weighted the communications aspect of C^3, to the detriment of not properly addressing the command and control (C^2) aspects, i.e, too often only seeking technological fixes to hard large-scale problems. For example, the Soviets give much weight to C^2, and they have structured tabular decision-aids dispersed through their levels of command. However, their relatively rigid political mind-set has fostered the development of these tables by using quasi-linear, essentially deterministic mathematical models fitted to operations data. If we use modern methods of nonlinear stochastic mathematical modeling, using data gained from operations as well as from more advanced computer models baselined to actual operations, then we can greatly increase our C^2 advantage.

C. NEED FOR MATHEMATICAL MODELS OF AGGREGATED COMBAT DATA

The reasons for seeking mathematical models of exercise data correspond to the same reasons for seeking mathematical models of computer models, as described in the previous section.

These mathematical models can be used to approach comparison of computer models, e.g., JANUS(T), to exercise data, e.g., from NTC. I.e., similar mathematical expressions must describe to similar mechanisms, whether they exist in the computer models or in the exercise data. Only if computer models such as JANUS(T) can be favorably compared to NTC data, can these wargames provide reliable pre- and post-exercise training for NTC commanders.

Such models also need to be developed at battalion-level, to drive corps- and theater-level models which will require highly aggregated models to run in real-time. For example, we expect good models of battalion-level combat to be nonlinear and stochastic. However, such models are too mathematically complex to drive higher echelon models, especially in real-time for wargaming. After this mathematical model is developed, only then is it correct and reasonable to determine decision rules, e.g., describing bifurcation of trends of the fitted mathematical distributions, and to linearize distributions in ‘‘most likely’’ regions. In this manner, the most salient features of properly fitted battalion-level models can drive higher echelon models.

Now, the task in this approach is to find the best mathematical model of each system, i.e., the computer-model system and the exercise-data system. Then, the best mathematical models for each can be compared.

D. MODELS VERSUS REALITY

It must be stated that there are still many problems faced by all computer models of combat, which must be solved before they can be accepted as models of reality.

For example, a very basic problem exists in the quality of acquisition algorithms, i.e., how to construct an algorithm that realistically portrays human attention (pre-attentive as well as selective) and perception, under various combat and weather conditions, night versus day, etc. The influence of attention and perception on complex physical [6-8] and mental tasks [9,10] has received considerable attention by one of the authors (LI). Currently, the best combat computer models treat acquisition as serial and logical processes, whereas the human brain acquires data by parallel and associative processes. Therefore, the inclusion of human players in multiple runs of similar scenarios is essential, if a probabilistic mathematical model is to be developed to model exercise data such as that obtained from NTC.

Presently, line of sight (LOS) algorithms seem to be the most costly time factor in running JANUS(T) computer models. Even if more realistic acquisition algorithms are developed, they must be tailored to the needs of real-time computer models if they are to be used in wargaming and in training.

Similarly, in order to develop a computer model of NTC exercises, to perform the comparison approach of this project, an acquisition model must be supplemented by existing algorithms within the computer model. This is a very difficult problem, requiring subjective, albeit expert, judgment.

E. INDIVIDUAL PERFORMANCE

It is clear that the individual performance is extremely important in combat [11], ranging in scale from battle management of the commander, to battle leadership of sub-commanders, to the degree of participation of individual units, to the more subtle degradation of units performing critical tasks.

Our analyses of NTC data concludes that data collected to date is not sufficient to accurately statistically judge individual performance across these scales. However, we do believe that this data is sufficient to analyze battle management, perhaps battle leadership in some cases. This is essential if we are to statistically compare JANUS(T) to NTC, and thereby prepare a what-if capability for JANUS(T) to augment NTC training.

It is important to recognize and emphasize the necessity of improving data collection at NTC, to permit complementary analyses of human factors at finer scales than our statistical approach permits.

F. RATIONALE

Yes, it would best to train for real combat always in full theater-level exercises, but that is not practical. Yes, next, it would be best to train on individual simulators, e.g., SIMNET, or perhaps with a subset of individual simulators complemented with computer simulations with individual man and system resolution. However, at the corps or theater levels, this also does not seem economically feasible. Even if it were, in order to understand the role of the human in the loop at several levels of command, still some aggregate models would need to be developed to transfer information between various spatial-temporal and system scales, the same problem faced by other physical, biological and societal systems.

These problems in such nonlinear nonequilibrium systems are ‘‘solved’’ (accommodated) by having new entities/languages developed at these disparate scales in order to efficiently pass information back and forth [12]. This is quite different from the nature of quasi-equilibrium quasi-linear systems, where a thermodynamic or cybernetic approach is possible.

The concept is to alter the database of the computer simulations to reflect exercise missions, fit this data to mathematical models and thereby compare various mathematical and computer models. Exercises are not true combat, but baselining a model to exercises for purposes of extrapolation to intended combat (with the proper database) is certainly preferable to not bothering to baseline models to any reality.

II. TECHNICAL BACKGROUND: GENERAL

A. PROBLEMS IN LANCHESTER THEORY

Quasi-linear deterministic mathematical modeling is not only a popular theoretical occupation, but many wargames, e.g., JTLS (Joint Theater Level Simulation), use such equations as the primary algorithm to drive the interactions between opposing forces.

In its simplest form, this kind of mathematical modeling is known as Lanchester theory:

$$\dot{r} = dr/dt = x_r b + y_r r b$$

$$\dot{b} = db/dt = x_b r + y_b b r \qquad (1)$$

where r and b represent Red and Blue variables, and the x's and y's are parameters which somehow should be fit to actual data.

It is well known, or should be well known, that it is notoriously difficult, if not impossible, to use Eq. (1) to mathematically model any real data with any reasonable degree of precision. These equations perhaps are useful to discuss some gross systematics, but it is hard to believe that, for example, a procurement decision involving billions of dollars of national resources would hinge on mathematical models dependent on Eq. (1).

Some investigators have gone further, and amassed historical data to claim that there is absolutely no foundation for believing that Eq. (1) has anything to do with reality [13].

However, although there is some truth to the above criticisms, the above conclusions do not sit comfortably with other vast stores of human experience. Indeed, this controversy is just one example that supports the necessity of having human intervention in the best of C^2 plans, no matter how (seemingly) sophisticated analysis supports conclusions contrary to human judgement [11]. I.e., when dealing with a dynamic complex system, intuition and analysis must join together to forge acceptable solutions.

The use of historical data, to disclaim any truth to the validity of Eq. (1), is in itself the use of inappropriate analysis [14]. The aggregation of solitary trajectories from many different stochastic combat scenarios does not necessarily form any kind of probability distribution upon which to make statistical judgements. Two combat scenarios, that differ in even only several variables, realistically are going to be quite different scenarios, not least because the very nature of nonlinear nonequilibrium (open) competitive systems is to have opposing sides pressed to their extreme, not to their average, capabilities. Given some ‘‘maneuvering room’’ to distort states of an open system, indeed these states will be distorted to new ‘‘favorable’’ values.

Therefore, as understood from experience in simulating physics systems, many trajectories of the ‘‘same’’ stochastic system must be aggregated before a sensible resolution of averages and fluctuations can be ascertained. Given two scenarios that differ in one parameter, and given a sufficient number of trajectories of each scenario, then the sensitivity to changes of a ‘‘reasonable’’ algebraic function to this parameter can offer some analytic input into decisions involving the use of this parameter in combat scenarios.

B. EMPIRICAL DATA

Therefore, there are two remaining issues to be resolved. The first is to find a database of a sufficient number of trajectories of the ‘‘same’’ system, upon which mathematical models can be built. The second is to forge an effective approach to mathematically model this data.

The numerous battalion cycles of exercises at the NTC can provide more trajectories of similar large-scale combat scenarios than any other source.

However, typical of exercises, whose purpose is to train and not necessarily to provide data serving analyses, this data is quite ‘‘dirty.’’ [15] Some problems specific to exercises would not occur in actual combat. There is a tremendous amount of several kinds of data, machine derived as well as derived from human observers in the form of ‘‘take-home packages.’’ [16,17]

The above is not meant to be unconstructive criticism of exercises at NTC. Quite the contrary, while respecting the sensitivity of this data, objective analyses for this project require a complete understanding of these problems.

C. MATHEMATICAL MODELING

This brings us to the next issue. What is a ‘‘reasonable’’ mathematical modeling approach?

It is reasonable to at least tentatively accept the experience of many commanders, whose intuitions have developed to think in terms of Eq. (1). Then, the problem seems to be that the degree of their quantitative, not qualitative, insights is insufficient to detail many combat scenarios. This then becomes the job of analysis, and explicates the purpose as well as the analytic task of mathematically modeling combat data. I.e., a good mathematical model must fit the data, and also be useful as a decision-aid to the commander and decision-maker.

Therefore, we can approach this problem by considering Eq. (1) as some kind of zeroth order approximation to reality.

In the late 1970’s, mathematical physicists discovered that they could develop statistical mechanical theories from algebraic functional forms

$$\dot{r} = f_r(r, b) + \sum_i \hat{g}_r^i(r, b)\eta_i$$

$$\dot{b} = f_b(b, r) + \sum_i \hat{g}_b^i(b, r)\eta_i \tag{2}$$

where the $\hat{g}$’s and f’s are general nonlinear algebraic functions of the variables r and b [18-23]. The f’s are referred to as the (deterministic) drifts, and the square of the $\hat{g}$’s are related to the diffusions (fluctuations). In fact, the statistical mechanics can be developed for any number of variables, not just two. The η’s are sources of Gaussian-Markovian noise, often referred to as ‘‘white noise.’’ The inclusion of the $\hat{g}$’s, called ‘‘multiplicative’’ noise, recently has been shown to very well mathematically and physically model other forms of noise, e.g., shot noise, colored noise, dichotomic noise [24-26]. Eq. (1)

is a special case of this generalized set of equations, with bilinear drift and mathematically zero noise. At this time, certainly the proper inclusion of multiplicative noise, using parameters fit to data to mathematically model general sources of noise, is preferable to improper inclusion or exclusion of any noise in combat models.

The ability to include many variables also permits a ‘‘field theory’’ to be developed, e.g., to have sets of (r, b) variables (and their rate equations) at many grid points, thereby permitting the exploration of spatial-temporal patterns in r and b variables. This gives the possibility of mathematically modeling the dynamic interactions across a large terrain.

D. SUPPORT FOR PRESENT MATHEMATICAL MODELING APPROACH

These new methods of nonlinear statistical mechanics only recently have been applied to complex large-scale physical problems, demonstrating that empirical data can be described by the use of these algebraic functional forms. Success was gained for large-scale systems in neuroscience, in a series of papers on statistical mechanics of neocortical interactions (SMNI) [27-38], and in nuclear physics [39-42]. I have proposed that these methods be used for problems in C^3 [12,24,36,43-45].

Thus, now we can investigate various choices of f’s and $\hat{g}$’s to see if algebraic functional forms close to the Lanchester forms can actually fit the data. In physics, this is the standard phenomenological approach to discovering and encoding knowledge and empirical data, i.e., fitting algebraic functional forms which lend themselves to physical interpretation. This gives more confidence when extrapolating to new scenarios, exactly the issue in building confidence in combat computer models.

The utility of these algebraic functional forms in Eq. (2) goes further beyond their being able to fit sets of data. There is an equivalent representation to Eq. (2), called a ‘‘path-integral’’ representation for the long-time probability distribution of the variables. This short-time probability distribution is driven by a ‘‘Lagrangian,’’ which can be thought of as a dynamic algebraic ‘‘cost’’ function. The path-integral representation for the long-time distribution possesses a variational principle, which means that simple graphs of the algebraic cost-function give a correct intuitive view of the most likely states of the variables, and of their variances. Like a ball bouncing about a terrain of hills and valleys, one can quickly visualize the nature of dynamically unfolding r and b states.

Especially because we are trying to mathematically model sparse and poor data, different drift and diffusion algebraic functions can give approximately the same algebraic cost-function when fitting short-time probability distributions to data. The calculation of long-time distributions permits a clear choice of the best algebraic functions, i.e., those which best follow the data through a predetermined epoch of battle. Thus, dynamic physical mechanisms, beyond simple ‘‘line’’ and ‘‘area’’ firing terms, can be identified. Afterwards, if there are closely competitive algebraic functions, they can be more precisely assessed by calculating higher algebraic correlation functions from the probability distribution.

It must be clearly stated that, like any other theory applied to a complex system, these methods have their limitations, and they are not a panacea for all systems. For example, probability theory itself is not a complete description when applied to categories of subjective ‘‘possibilities’’ of information [46,47]. Other non-stochastic issues are likely appropriate for determining other types of causal relationships, e.g., the importance of reconnaissance to success of missions [17]. These statistical mechanical methods appear to be appropriate for comparing these stochastic large-scale combat JANUS(T) and NTC systems. The details of our studies will help to determine the correctness of this premise.

III. TECHNICAL APPROACH

A. COMPLEXITY OF TYPICAL PROBLEMS

There are variables, spatial dimensions, and parameters that must be processed by such calculations. Typically researchers have considered only a few variables, e.g., one or two, in one or two dimensions, with several parameters; or, they have considered limiting cases of huge/infinite number of variables/dimensions. These problems require breaking new ground into the nonlinear nonequilibrium stochastic realm of 10, 20 or 30 dynamic variables. This number is barely large enough to give reliable analysis/aids to decision-makers, yet barely small enough to be able to process good scientific calculations. We must avoid handling too many variables which leads quickly to data overload of machines as well as of humans, and we must avoid doing too simplistic modeling which is at best unreliable for complex systems.

To better illustrate some of these points, consider the scenario mapped out in time and in the two spatial dimensions, where the spatial dimensions are coarsely grained into a three by three grid of cells, e.g., labeled from (1,1) through (3,3). (See Fig. 2.) Consider that Blue forces have strategically set themselves into a defensive posture on the West German border at some site, here located in the middle left column cell. Consider Red forces have strategically set themselves into an offensive posture on the East German border, here located throughout the right column of three cells. Consider that all microscopic stochastic algorithms defining each person's and machine's interactions at given distances and velocities have already been programmed into the simulation.

Figure 2

There will be long-ranged interactions, e.g., via artillery, as well as short-ranged ground interactions. Each cell defines an independent set of Blue and Red stochastic mesoscopic variables. E.g., in cell (2,3) there may be two kinds of Blue forces/weapons, etc., interacting with the other Blue and Red variables at all cells.

To appreciate the magnitude of the problem being presented, consider a relatively simple mathematical model of Blue and Red, each possessing only one type of force/weapon. Further simplify the problem by considering that the rate of change of each variable in each cell is driven by time-independent algebraic functions: a drift-force with terms proportional to Red attrition, and terms proportional to Red attrition multiplied by Blue attrition; a multiplicative noise term composed of a constant background, and terms proportional to Red or Blue attrition. I.e., consider four intra-cell parameters per Blue and Red variable per cell. In addition, for simplicity, consider only nearest-neighbor (NN) interactions between cells, effected by adding linear terms to each drift proportional to Red and Blue attrition. Thus each NN requires an additional 4 parameters per cell. I.e., each corner cell has 2 NN, the middle cell has 4 NN, the others have 3 NN each. This then defines a 168-dimensional parameter-space of coefficients in a mathematical model Lagrangian defined by an 18-dimensional variable-space in two spatial dimensions, to be fit by a maximum-likelihood algebraic function of the short-time probability distribution of the variables to combat simulation data. Of course, for well-known scenarios, intuition gained by working with experts in combat analysis will greatly reduce the number of meaningful parameters to be considered.

The cells serve to aggregate the appropriate mesoscopic variables, here to be considered as the spatial-temporal attrition of Blue and Red units during the course of the battle. This then describes a classic pattern-recognition problem, to describe the spatial-temporal evolution of these variables.

Even for just two cells with two stochastic variables, the number of parameters can be quite large:

Cell 1:

$$\dot{r}_1 = x^{r_1}_{b_1} b_1 + y^{r_1}_{b_1 r_1} b_1 r_1 + z^{r_1} \eta_{r_1} + z'^{r_1} r_1 \eta'_{r_1} + x^{r_1}_{b_2} b_2 + x^{r_1}_{r_2} r_2$$

$$\dot{b}_1 = x^{b_1}_{r_1} r_1 + y^{b_1}_{r_1 b_1} r_1 b_1 + z^{b_1} \eta_{b_1} + z'^{b_1} b_1 \eta'_{b_1} + x^{b_1}_{b_2} b_2 + x^{b_1}_{r_2} r_2 \qquad (3)$$

Cell 2:

$$\dot{r}_2 = x^{r_2}_{b_2} b_2 + y^{r_2}_{b_2 r_2} b_2 r_2 + z^{r_2} \eta_{r_2} + z'^{r_2} r_2 \eta'_{r_2} + x^{r_2}_{b_1} b_1 + x^{r_2}_{r_1} r_1$$

$$\dot{b}_2 = x^{b_2}_{r_2} r_2 + y^{b_2}_{r_2 b_2} r_2 b_2 + z^{b_2} \eta_{b_2} + z'^{b_2} b_2 \eta'_{b_2} + x^{b_2}_{b_1} b_1 + x^{b_2}_{r_1} r_1 \quad (4)$$

where

$r_{1,2}$: red attrition, number of casualties in cell 1,2

$b_{1,2}$: blue attrition, number of casualties in cell 1,2

$\eta_{1,2}$: uncertainty, white noise in cell 1,2

$\eta'_{1,2}$: uncertainty, multiplicative noise in cell 1,2

x, y, z, z': parameters to be fit to data

$x_{1,2}$-terms: attrition due to direct ‘‘line’’ firing

$y_{1,2}$-terms: attrition due to ‘‘area’’ firing

$z_{1,2}$, $z'_{1,2}$-terms: uncertainty in physics and C^3 information

Note that, in general, the x’s, y’s and z’s may be time-dependent, but in this first set of studies, they are taken as constants. Although this statistical mechanics approach can process this time-dependence, it greatly adds to the resources necessary to fit the data.

The z' terms include the interesting physical mechanism, describing the uncertainty in attrition during each short interval of time as being proportional to the total force at the beginning of the interval. This effectively introduces a highly nonlinear log-normal behavior, but presents no additional problems for our quite general calculational procedures.

B. GAUSSIAN-MARKOVIAN ANALYSES

As discussed previously [24], the mathematical representation most familiar to other modelers is a system of stochastic rate equations, often referred to as Langevin equations. From the Langevin equations, other models may be derived, such as the times-series model and the Kalman filter method of control theory. However, in the process of this transformation, the Markovian description typically is lost by projection onto a smaller state space [48,49]. This work only considers multiplicative Gaussian noise, including the limit of weak colored noise [25]. These methods are not conveniently used for other sources of noise, e.g., Poisson processes or Bernoulli processes. It remains to be seen if multiplicative noise can emulate these processes in the empirical ranges of interest, in some reasonable limits [26]. At this time, certainly the proper inclusion of multiplicative noise, using parameters fit to data to model general sources of noise, is preferable to improper inclusion or exclusion of any noise.

These mesoscopic functional forms and their coefficients-parameters can be fit to real empirical data, or at least to simulated data of real systems, to develop a time-dependent multivariable probability distribution defining the mesoscopic scale.

C. METHODOLOGY

Model development. Consider a scenario taken from a JANUS(T) replication of NTC: two Red systems, RT (Red tanks) and $RBMP$, and three Blue systems, BT, $BAPC$ and $BTOW$, where RT specifies the number of Red tanks at a given time t, etc. Consider the kills suffered by BT, ΔBT, e.g., within a time epoch $\Delta t \approx 5$ minutes:

$$\frac{\Delta BT}{\Delta t} \equiv \dot{BT} = x^{BT}_{RT} RT + y^{BT}_{RT} RT\ BT + x^{BT}_{RBMP} RBMP + y^{BT}_{RBMP} RBMP\ BT$$

$$+ z^{BT}_{BT} BT \eta^{BT}_{BT} + z^{BT}_{RT} \eta^{BT}_{RT} + z^{BT}_{RBMP} \eta^{BT}_{RBMP} \tag{5}$$

where the η's represent sources of (white) noise (in the Ito prepoint discretization). Here, the x terms represent attrition due to point fire; the y terms represent attrition due to area fire; the diagonal z term (z^{BT}_{BT}) represents uncertainty associated with the *target* BT, and the off-diagonal z terms represent uncertainty associated with the *shooters* RT and $RBMP$. The x's and y's are constrained such that each term is bounded by the mean of the Killer-Victim Scoreboard (KVS), averaged over all time and trajectories of similar scenarios; similarly, each z term is constrained to be bounded by the variance of the KVS.

Note that the functional forms chosen are consistent with current perceptions of aggregated large-scale combat. I.e., these forms reflect point and area firing; the noise terms are taken to be log-normal (multiplicative) noise for the diagonal terms and additive noise for the off-diagonal terms. The methodology presented here can accommodate any other nonlinear functional forms, and any other variables which can be reasonably represented by such rate equations, e.g., expenditures of ammunition or bytes of communication [12]. Variables which cannot be so represented, e.g., terrain, C^3, weather, etc., must be considered as ‘‘super-variables’’ which specify the overall context for the above set of rate equations.

Equations similar to the $\dot{BT}$ equation are also written for $\dot{RT}$, $\dot{RBMP}$, $\dot{BAPC}$, and $\dot{BTOW}$. Only x's and y's which reflect possible non-zero entries in the KVS are free to be used for the fitting procedure. For example, since JANUS(T) does not permit direct-fire fratricide, such terms are set to zero. Non-diagonal noise terms give rise to correlations in the covariance matrix. Thus, we have

$$M^G = \{RT, RBMP, BT, BAPC, BTOW\}$$

$$\dot{M}^G = g^G + \sum_i \hat{g}_i^G \eta^i$$

$$\hat{g}_i = \begin{cases} z_i^G M^G \ , \ i = G \\ z_i^G \ , \quad\quad i \neq G \end{cases} \tag{6}$$

Fitting parameters. These five coupled stochastic differential equations can be represented equivalently by a short-time conditional probability distribution, P, in terms of a Lagrangian, L:

$$P(R.\,, B.\,; t + \Delta t | R.\,, B.\,; t) = \frac{1}{(2\pi \Delta t)^{5/2} \sigma^{1/2}} \exp(-L \Delta t) \tag{7}$$

where σ is the determinant of the inverse of the covariance matrix, the metric matrix of this space, R.

represents $\{RT, RBMP\}$, and $B.$ represents $\{BT, BAPC, BTOW\}$. (Here, the prepoint discretization is used, which hides the Riemannian corrections explicit in the midpoint discretizatized Feynman Lagrangian; only the latter representation possesses a variational principle useful for arbitrary noise.)

This defines a scalar ‘‘dynamic cost function,’’ $C(x, y, z)$,

$$C(x, y, z) = L\Delta t + \frac{5}{2}\ln(2\pi\Delta t) + \frac{1}{2}\ln\sigma \tag{8}$$

which can be used with the Very Fast Simulated Re-Annealing (VFR) algorithm [50] to find the (statistically) best fit of $\{x, y, z\}$ to the data.

The form for the Lagrangian, L, and the determinant of the metric, σ, to be used for the cost function C, using Table 1, is:

$$L = \sum_G \sum_{G'} \frac{(\dot{M}^G - g^G)(\dot{M}^{G'} - g^{G'})}{2g^{GG'}}$$

$$\sigma = \det(g_{GG'}) \,,\; (g_{GG'}) = (g^{GG'})^{-1}$$

$$g^{GG'} = \sum_i \hat{g}_i^G \hat{g}_i^{G'} \tag{9}$$

Generated choices for $\{x, y, z\}$ are constrained by empirical (taken from exercises or from computer simulations of these exercises) KVS:

$$g^G(t) \le n^G < \Delta M^G(t) >$$

$$\hat{g}_i^G(t) \le n_i^G \mathbf{[}< \mathbf{(}\Delta M^G(t)\mathbf{)}^2 >\mathbf{]}^{1/2} \tag{10}$$

where n^G and n_i^G are the number of terms in g^G and $\hat{g}_i^G$, resp., and averages, $< . >$, are taken over all time epochs and trajectories of similar scenarios.

The Lagrangian representation makes it more clear how the variables in are to be viewed statistically. Furthermore, it is clear how to generalize these equations to spatial dimensions [43]. Then the stochastic variables are the attrition variables as a function of space and time; a true ‘‘field’’ representation is required.

Choosing a model. If there are competing mathematical forms, then it is advantageous to utilize the path-integral to calculate the long-time evolution of P [12]. Some limited experience has demonstrated that, since P is exponentially sensitive to changes in L, the long-time correlations derived from theory, measured against the empirical data, is a viable and expedient way of rejecting models not in accord with empirical evidence.

Note that the use of the path integral is *a posteriori* to the short-time fitting process, and is a subsidiary physical constraint on the mathematical models to judge their internal soundness and suitability for attempts to extrapolate to other scenarios.

Combat power scores. After the $\{ x, y, z, \}$ are fit to the data, and a mathematical model is selected, another fit can be superimposed to find the effective ‘‘combat scores,’’ defined here as the relative contribution of each system to the specific class of scenarios in question. Using a fundamental property of probability distributions, a probability distribution $P_A(q)$ of aggregated variables $q_1 + q_2$ can be obtained from the probability distribution for $P(q_1, q_2)$:

$$P_A(q = q_1 + q_2) = \int dq_1\, dq_2\, P(q_1, q_2)\delta(q - q_1 - q_2) \tag{11}$$

where $\delta(.)$ is the Dirac delta function.

Thus, we calculate the aggregated conditional probability

$$P_A(r, b; t + \Delta t | R., B.; t) = \int dRT\ dRBMP\ dBT\ dBAPC\ dBTOW$$

$$\times P(R., B.; t + \Delta t | R., B.; t)$$

$$\times\delta(r - w^r_{RT} RT - w^r_{RBMP} RBMP)$$

$$\times\delta(b - w^b_{RT} BT - w^b_{BAPC} BAPC - w^b_{BTOW} BTOW) \qquad (12)$$

where the w's represent the desired combat scores. For the first approach, it is worth trying doing the integral just over the variables at time $t + \Delta t$ to determine the w's. The dimension of the space over which the integral must be performed is effectively reduced by two, using the Dirac delta functions. Usually two more reductions can be made by performing two Gaussian integrals (if the covariance matrix is not too complex). As the remaining integrals likely must be done numerically, advantage can be taken of the ‘‘Boltzmann’’ form of the Lagrangian, and importance-sampling Monte Carlo routines can be used. This is particularly useful for problems with many more variables than the NTC example given here. Thus, after the $\{x, y, z\}$ have been fitted, the new parameters $\{w\}$ can be fit the data by maximizing the cost function $C'(w)$ using VFR,

$$C'(w) = -\ln P_A \qquad (13)$$

This second fitting procedure might require much CPU time, and therefore should be done only if the w's are truly of great interest. Of course, once these calculations are performed, say at division level for several classes of combat, then the theater-level models can enjoy real-time processing with input at least interfaced with empirical data.

IV. OUTLINE OF MATHEMATICAL METHODOLOGY

A. ONE VARIABLE, ONE CELL

The Langevin Rate-Equation exhibits a generalized Lanchester equation, wherein drifts and multiplicative noise can be arbitrarily nonlinear functions.

$$M(t + \Delta t) - M(t) \sim \Delta t\ f[M(t)]$$

$$\dot{M} = \frac{dM}{dt} \sim f$$

$$\dot{M} = f + \hat{g}\eta$$

$$< \eta(t) >_\eta = 0$$

$$< \eta(t)\eta(t') >_\eta = \delta(t - t') \quad (14)$$

$\eta(t)$ represents ‘‘white noise.’’

The Diffusion Equation is another equivalent representation of Langevin equations. This is also called the Fokker-Planck equation. The first moment ‘‘drift’’ is identified as f, and the second moment ‘‘diffusion,’’ the variance, is identified as $\hat{g}^2$.

$$\frac{\partial P}{\partial t} = \frac{\partial(-fP)}{\partial M} + \frac{1}{2}\frac{\partial^2(\hat{g}^2 P)}{\partial M^2} \quad (15)$$

The Path-Integral Lagrangian represents yet another equivalent representation of Langevin equations, but one that can offer superior decision aids to the commander. Recently it has been demonstrated that the drift and diffusion, in addition to possibly being quite general nonlinear functions of the independent variables and of time explicitly, may also be explicit functions of the distribution P itself, and possesses path integral solutions [51]. This property lends these methods to modeling strategies within combat scenarios. The short-time conditional probability, of measuring $M(t + \Delta t)$ at time $t + \Delta t$ given $M(t)$ at time t, is given by

$$P[M_{t+\Delta t}|M_t] = (2\pi \hat{g}^2 \Delta t)^{-1/2} \exp(-\Delta t L)$$

$$L = (\dot{M} - f)^2/(2\hat{g}^2)$$

$$P[M_t|M_{t_0}] = \int \cdots \int dM_{t-\Delta t} dM_{t-2\Delta t} \cdots dM_{t_0+\Delta t}$$

$$\times P[M_t|M_{t-\Delta t}]P[M_{t-\Delta t}|M_{t-2\Delta t}]$$

$$\times \cdots P[M_{t_0+\Delta t}|M_{t_0}]$$

$$P[M_t|M_{t_0}] = \int \cdots \int \underset{\sim}{D} M \exp(-\sum_{s=0}^{u} \Delta t L_s)$$

$$\underset{\sim}{D} M = (2\pi \hat{g}_0^2 \Delta t)^{-1/2} \prod_{s=1}^{u} (2\pi \hat{g}_s^2 \Delta t)^{-1/2} dM_s$$

$$\int dM_s \to \sum_{\alpha=1}^{N} \Delta M_{\alpha s} , M_0 = M_{t_0} , M_{u+1} = M_t \tag{16}$$

where L_s is the Lagrangian at time $t_s = t_0 + s\Delta t$.

B. MANY NONLINEAR VARIABLES

Now, consider a multivariate system, again in just one cell, but with the multivariate variance a general nonlinear function of the variables. Similar equations previously were used to develop the mesocolumn for the neocortical system. The Einstein summation convention helps to compact the equations, whereby repeated indices in factors are to be summed over.

The Itô (prepoint) discretization for a system of stochastic differential equations is defined by

$$\bar{t}_s \in [t_s, t_s + \Delta t]$$

$$M(\bar{t}_s) = M(t_s)$$

$$\dot{M}(\bar{t}_s) = M(t_{s+1}) - M(t_s) \tag{17}$$

The stochastic equations are then written as

$$\dot{M}^G = f^G + \hat{g}^G_i \eta^i$$

$$i = 1, \cdots, \Xi$$

$$G = 1, \cdots, \Theta \tag{18}$$

The operator ordering (of the $\partial/\partial M^G$ operators) in the Fokker-Planck equation corresponding to this discretization is

$$\frac{\partial P}{\partial t} = VP + \frac{\partial(-g^G P)}{\partial M^G} + \frac{1}{2}\frac{\partial^2(g^{GG'} P)}{\partial M^G \partial M^{G'}}$$

$$g^G = f^G + \frac{1}{2}\hat{g}^{G'}_i \frac{\partial \hat{g}^G_i}{\partial M^{G'}}$$

$$g^{GG'} = \hat{g}^G_i \hat{g}^{G'}_i \tag{19}$$

The Lagrangian corresponding to this Fokker-Planck and set of Langevin equations may be written in the Stratonovich (midpoint) representation, corresponding to

$$M(\bar{t}_s) = \frac{1}{2}\,[M(t_{s+1}) + M(t_s)] \qquad (20)$$

This discretization defines a Feynman Lagrangian L which possesses a variational principle, and which explicitly portrays the underlying Riemannian geometry induced by the metric tensor $g_{GG'}$.

$$P = \int \cdots \int \underset{\sim}{D}M \exp(-\sum_{s=0}^{u} \Delta t L_s)$$

$$\underset{\sim}{D}M = g_{0_+}^{1/2} (2\pi\Delta t)^{-1/2} \prod_{s=1}^{u} g_{s_+}^{1/2} \prod_{G=1}^{\Theta} (2\pi\Delta t)^{-1/2} dM_s^G$$

$$\int dM_s^G \to \sum_{\alpha=1}^{N^G} \Delta M_{\alpha s}^G \,,\; M_0^G = M_{t_0}^G \,,\; M_{u+1}^G = M_t^G$$

$$L = \frac{1}{2}(\dot{M}^G - h^G) g_{GG'} (\dot{M}^{G'} - h^{G'}) + \frac{1}{2} h^G{}_{;G} + R/6 - V$$

$$[\cdots]_{,G} = \frac{\partial[\cdots]}{\partial M^G}$$

$$h^G = g^G - \frac{1}{2} g^{-1/2} (g^{1/2} g^{GG'})_{,G'}$$

$$g_{GG'} = (g^{GG'})^{-1}$$

$$g_s[M^G(\bar{t}_s), \bar{t}_s] = \det(g_{GG'})_s \,,\; g_{s_+} = g_s[M_{s+1}^G, \bar{t}_s]$$

$$h^G{}_{;G} = h^G_{,G} + \Gamma^F_{GF} h^G = g^{-1/2} (g^{1/2} h^G)_{,G}$$

$$\Gamma^F_{JK} \equiv g^{LF}[JK, L] = g^{LF}(g_{JL,K} + g_{KL,J} - g_{JK,L})$$

$$R = g^{JL} R_{JL} = g^{JL} g^{JK} R_{FJKL}$$

$$R_{FJKL} = \frac{1}{2} (g_{FK,JL} - g_{JK,FL} - g_{FL,JK} + g_{JL,FK})$$

$$+ g_{MN} (\Gamma^M_{FK} \Gamma^N_{JL} - \Gamma^M_{FL} \Gamma^N_{JK}) \tag{21}$$

Because of the presence of multiplicative noise, the Langevin system differs in its Itô (prepoint) and Stratonovich (midpoint) discretizations. The midpoint-discretized covariant description, in terms of the Feynman Lagrangian, is defined such that (arbitrary) fluctuations occur about solutions to the Euler-Lagrange variational equations. In contrast, the usual Itô and corresponding Stratonovich discretizations are defined such that the path integral reduces to the Fokker-Planck equation in the weak-noise limit. The term $R/6$ in the Feynman Lagrangian includes a contribution of $R/12$ from the WKB approximation to the same order of $(\Delta t)^{3/2}$ [22].

C. MANY CELLS

Now, consider the generalization to many cells. Similar mathematics is used to describe large regions of neocortex. In the absence of any further information about the system, this increases the number of variables, from the set $\{G\}$ to the set $\{G, \nu\}$. The nonlinear variances require that the discretization be specified in order to be consistent with a given diffusion partial differential equation. The Feynman Lagrangian $\tilde{L}$ given here, in a covariant midpoint discretization, satisfies a variational principle for arbitrary noise.

$$\tilde{P} = \int \cdots \int \underset{\sim}{D} \tilde{M} \exp\left(- \sum_{s=0}^{u} \Delta t \tilde{L}_s\right)$$

$$\tilde{M} = \{ M_s^{G\nu} | G = 1, \cdots \Theta; \nu = 1, \cdots, \Lambda; s = 1, \cdots, u \}$$

$$\underline{D}\tilde{M} = \tilde{g}_{0_+}^{1/2}(2\pi\Delta t)^{-1/2} \prod_{s=1}^{u} \tilde{g}_{s_+}^{1/2} \prod_{G=1}^{\Theta} \prod_{\nu=1}^{\Lambda} (2\pi\Delta t)^{-1/2} dM_s^{G\nu}$$

$$\tilde{g}_{G\nu}(\bar{t}_s), \bar{t}_s] = \det(g_{GG'\nu\nu'})_s \, , \tilde{g}_{s_+} = \tilde{g}_s[M_{s+1}^{G\nu}, \bar{t}_s]$$

$$\int dM_s^{G\nu} \to \sum_{\alpha=1}^{N^G} \Delta M_{\alpha s}^{G\nu} \, , M_0^{G\nu} = M_{t_0}^{G\nu} \, , M_{u+1}^{G\nu} = M_t^{G\nu}$$

$$\tilde{L} = \frac{1}{2}(\dot{M}^{G\nu} - h^{G\nu})g_{GG'\nu\nu'}(\dot{M}^{G'\nu'} - h^{G'\nu'})$$

$$+ \frac{1}{2} h^{G\nu}{}_{;G\nu} + \tilde{R}/6 - \tilde{V}$$

$$M^G(\bar{t}_s) = \frac{1}{2}(M_{s+1}^G + M_s^G) \, , \dot{M}^G(\bar{t}_s) = (M_{s+1}^G - M_s^G)/\Delta t$$

$$\bar{t}_s = t_s + \Delta t/2 \tag{22}$$

A different prepoint discretization for the same probability distribution $\tilde{P}$, gives a much simpler algebraic form, but the Lagrangian $\tilde{L}'$ so specified does not satisfy a variational principle useful for moderate to large nosie. Still, this prepoint-discretized form has been quite useful in all systems examined thus far, as a first approximation.

D. FITTING THE INFORMATION IN THE LAGRANGIAN

The Lagrangian must be fitted to empirical data in two nested procedures: Within sets of trial functions, each set must have its parameters/coefficients fitted. Then the probability distribution, considered as a functional of its variables, can be used to describe the evolution of the system.

$$\tilde{L}' = \frac{1}{2}(\dot{M}^{G\nu} - g^{G\nu})g_{GG'\nu\nu'}(\dot{M}^{G'\nu'} - g^{G'\nu'}) - \tilde{V}$$

$$\dot{M}^{G\nu}(\bar{t}_s) = (M^{G\nu}_{s+1} - M^{G\nu}_s)/\Delta t \ , \ M^{G\nu}(\bar{t}_s) = M^{G\nu}_s$$

$$\bar{t}_s = t_s \ , \ \tilde{g}_{s_+} = \tilde{g}_s$$

$$\tilde{V} = \tilde{V}' + J_G M^G$$

$$\tilde{I} = -\int d\tilde{M}_t \tilde{P} \ln(\tilde{P}/\underline{\tilde{P}})$$

$$\text{NN:} \sum_\nu \sum_{\nu'} \rightarrow \sum_\nu \sum_{\nu_{\text{NN}}}$$

$$\tilde{V}' = \tilde{V}'' + Z_G(\nabla_{\nu_{\text{NN}}} \underline{M}^G) + Z_{GG'}(\nabla_{\nu_{\text{NN}}} \underline{M}^G)(\nabla_{\nu'_{\text{NN}}} \underline{M}^{G'}) + \cdots$$

$$g^G = X^G_{G'} \underline{M}^{G'} + X^G_{G'G''} \underline{M}^{G'} \underline{M}^{G''} + \cdots$$

$$g_{GG'} = Y_{GG'} + Y_{GG'G''} \underline{M}^{G''} + Y_{GG'G''G'''} \underline{M}^{G''} \underline{M}^{G'''} + \cdots$$

$$\underline{M}^{G\nu}_s = M^{G\nu}_s - \ll M^{G\nu}_s \gg \tag{23}$$

The X's and Y's also should be indexed with respect to the multiple minima. As mentioned previously, g^G and $g^{GG'}$ may also be explicit functions of the distribution P, enabling strategies to be modeled. I.e., the drifts and diffusion may be functions of the state-of-affairs.

Once the parameters $\{X, Y, Z, \ll M^{G\nu}_s \gg \}$, are fit, the theory is ready to track or predict. For many phenomenological models, where an underlying microscopic theory does not exist, probability distributions must be folded with statistical analyses of fitted parameters to determine a given system. Science is not only empiricism. Modeling and chunking of information is required, not only for aesthetics, but also to reduce required computational resources of brains as well as machines.

E. CURRENT STATE OF ALGORITHMS

Recently, two major computer codes have been developed, which are key tools for use of this approach to mathematically model combat data.

The first code fits short-time probability distributions to empirical data, using a most-likelihood technique on the Lagrangian. An algorithm of very fast simulated re-annealing has been developed to fit a empirical data to a theoretical cost function over a D-dimensional parameter space [50], adapting for varying sensitivities of parameters during the fit. The annealing schedule for the ‘‘temperatures’’ (artificial fluctuation parameters) T_i decrease exponentially in ‘‘time’’ (cycle-number of iterative process) k, i.e., $T_i = T_{i0} \exp(-c_i k^{1/D})$.

Heuristic arguments have been developed to demonstrate that this algorithm is faster than the fast Cauchy annealing [52], $T_i = T_0/k$, and much faster than Boltzmann annealing [53], $T_i = T_0/\ln k$. To be more specific, the kth estimate of parameter α^i,

$$\alpha^i_k \in [A_i, B_i] \tag{24}$$

is used with the random variable x^i to get the $k+1$th estimate,

$$\alpha^i_{k+1} = \alpha^i_k + x^i (B_i - A_i)$$

$$x^i \in [-1, 1] \tag{25}$$

Define the generating function

$$g_T(x) = \prod_{i=1}^{D} \frac{1}{2 \ln(1 + 1/T_i)(|x^i| + T_i)} \equiv \prod_{i=1}^{D} g^i_T(x^i)$$

$$T_i = T_{i0} \exp(-c_i k^{1/D}) \tag{26}$$

The cost-functions $\underline{L}$ we are exploring are of the form

$$h(M;\alpha) = \exp(-\underline{L}/T)$$

$$\underline{L} = L\Delta t + \frac{1}{2}\ln(2\pi\Delta t g_t^2) \qquad (27)$$

where L is a Lagrangian with dynamic variables $M(t)$, and parameter-coefficients α to be fit to data. g_t is the determinant of the metric.

The second code develops the long-time probability distribution from the Lagrangian fitted by the first code. A robust and accurate histogram path-integral algorithm to calculate the long-time probability distribution has been developed by one of us (MW) to handle nonlinear Lagrangians [51,54,55], including a two-variable code for additive and multiplicative cases. We are presently working to create a code to process several variables.

F. RELATIONSHIP TO OTHER COMBAT MODELING

Connections can be made to other attrition-driven models which require very simple algebraic forms to drive their computer models in real time. After fitting battalion-level data with our methodology, a combination of decision rules to determine space-time regions of maximum probability from the stochastic nonlinear fits, and local linearization of these probability peaks to Lanchester-type simpler algebraic forms, can satisfy these requirements. This application can be made to higher-echelon Army models, e.g., VIC (division to corps level), FORCEM (EAC to theater level), JTLS (theater level), JESS (division to corps level).

These methods also be used to implement C^3 models driven by attrition equations such as C3EVAL being developed by Institute for Defense Analyses for the J6−F office of JCS. This will permit connection of JANUS(T) to higher echelon C^3, and/or to degrade/jam communications between wargamers.

This Lagrangian approach to combat dynamics permits a quantitative assessment of concepts previously only loosely defined.

$$\text{``Momentum''} = \Pi^G = \frac{\partial L}{\partial(\partial M^G/\partial t)}$$

$$\text{``Mass''} = g_{GG'} = \frac{\partial L}{\partial(\partial M^G/\partial t)\partial(\partial M^{G'}/\partial t)}$$

$$\text{``Force''} = \frac{\partial L}{\partial M^G}$$

$$\text{``}F = ma\text{''}: \ \delta L = 0 = \frac{\partial L}{\partial M^G} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial M^G/\partial t)} \tag{28}$$

where M^G are the variables and L is the Lagrangian. These relationships are derived and are valid at each spatial point x of $M^G(x,t)$. Reduction to other math-physics modeling can be achieved after fitting realistic exercise and/or simulation data.

E.g., phase transitions can be investigated as bifurcation develops in path-integral calculations [56]. Investigations into the possibility of chaos can take advantage of using algebraic models fitted to data [57,58]. The development of chaos might present opportunities to induce chaos in opponents. This can be studied by examining the temporal folding of the path integral as it develops attractors in the presence of noise. Catastrophe theory can be used to study the critical region of a time-slice or static limit of Taylor-expanded/approximated Lagrangian [59].

Chaos and fractals can be directly investigated by directly sampling raw data [60]. However, algebraic models of data are very useful, sometimes necessary, when presented by sparse data, or when it is necessary to extrapolate to regions where no data is available. It should be noted that chaos is very difficult to separate from ‘‘classical’’ randomness for realistic systems. Only a few realistic systems are proved to be K-systems. Also, note that our path-integral approach serves to define just what ‘‘short time’’ is required such that discrete combat can be approximated by differential equations. This typically turns out to be on the order of a few minutes for battalion-brigade engagements.

Predator-prey biological models are interesting, but not necessarily relevant to combat studies. I.e., war is predator-predator, and a change in the sign of a term in a differential equation results in quite different solutions.

G. UNIT PERFORMANCE FROM JANUS(T) NTC SURROGATE MODEL

The best resolution presently available from NTC is at the company level. Our JANUS(T) what-if model can provide better resolution, at least statistically consistent with NTC data. E.g., we can distinguish between reconnaissance and active combatants, or between good shooters and poor shooters.

As an example, consider

$$\dot{b}_1 = f_{b_1}(b_1, b_2, r) + \hat{g}_{b_1}(b_1, b_2, r)\eta^i_{b_1}$$

$$\dot{b}_2 = f_{b_2}(b_1, b_2, r) + \hat{g}_{b_2}(b_1, b_2, r)\eta^i_{b_2}$$

$$\dot{r} = f_r(b_1, b_2, r) + \hat{g}_r(b_1, b_2, r)\eta^i_r$$

$$f_r = - x b_1 - y b_2 \qquad (29)$$

If $x > y$, then b_1 is a better shooter than b_2.

V. NTC PROTOTYPE MATHEMATICAL MODEL

The mathematical model comparison process develops separate mathematical models for both the computer simulation data and the exercise data, thereby permitting a common basis for quantitative comparison. (See Fig. 3.) Performing this task requires intimate knowledge of each system as well the mathematical tools described previously.

Figure 3

A. DESCRIPTION OF NTC

The U.S. Army National Training Center (NTC) is located at Fort Irwin, just outside of Barstow, California.

There have been about 1/4 million soldiers in 80 brigade rotations at NTC, at about the level of 2 battalion task forces (typically about 3500 soldiers and a battalion of 15 attack helicopters), which train against 2 opposing force (OPFOR) battalions resident at NTC. NTC comprises about 2500 km^2, but the current battlefield scenarios range over about 5 km linear spread, with a maximum lethality range of about 3 km. NTC is gearing up for full brigade-level exercises.

Observer-controllers (OC) are present down to about platoon level. A rotation will have three force-on-force missions and one live-fire mission. OPFOR platoon- and company-level personnel are trained as US Army soldiers; higher commanders practice Soviet doctrine and tactics. An OPFOR force typically has ~100 BMP’s and ~40 T72’s.

The primary purpose of data collection during an NTC mission is to patch together an after-action review (AAR) within a few hours after completion of a mission, giving feedback to a commander who typically must lead another mission soon afterwards. Data from the field, multiple integrated laser engagement system (MILES) devices, audio communications, OC’s, and stationary and mobile videocameras, is sent via relay stations back to a central command center where this all can be recorded, correlated and abstracted for the AAR. Within a couple of weeks afterwards, a written review is sent to commanders, as part of their NTC take-home package. It presently costs about 4 million dollars per NTC rotation, 1 million of which goes for this computer support.

There are 460 MILES transponders available for tanks for each battle. The ‘‘B’’ units have transponders, but most do not have transmitters to enable complete pairings of kills-targets to be made. (New MILES devices being implemented have transmitters which code their system identification, thereby greatly increasing the number or recordings of pairings.) Thus, MILES’s without transmitters cannot be tracked. Man-packs with B units enable these men to be tracked, but one man-pack can represent an aggregate of as much as 5 people.

B units send data to ‘‘A’’ stations (presently 48, though 68 can be accommodated), then collected by two ‘‘C’’ stations atop mountains, and sent through cables to central VAX’s forming a core instrumentation system (CIS).

There is a present limitation of 400 nodes in computer history for video tracking (but 500 nodes can be kept on tape). Therefore, about 200 Blue and 200 OPFOR units are tracked.

By varying the laser intensity and focusing parameters, a maximum laser-beam spread is achieved at the nominal range specified by the Army. A much narrower beam can reach as far as the maximum range. Focusing and attenuation properties of the laser beam makes these nominal and maximum ranges quite sharp, supposedly considerably less than several hundred meters under ideal environmental conditions. For example, a weapon might send out a code of 8 words (spaced apart by nsecs), 2 of which must register on a target to trigger the Monte Carlo routine to calculate a PK. Attenuation of the beam past its preset range means that it rapidly becomes unlikely that 2 words will survive to reach the target.

B. DESCRIPTION OF JANUS(T)

JANUS(T) is an interactive, two-sided, closed, stochastic, ground combat (recently expanded to air and naval combat as an extension of our present projects) computer simulation.

Interactive refers to the the fact that military analysts (players and controllers) make key complex decisions during the simulation, and directly react to key actions of the simulated combat forces. Two-sided (hence the name Janus of the Greek two-headed god) means that there are two opposing forces simultaneously being directed by two set of players. Closed means that the disposition of the enemy force is not completely known to the friendly forces. Stochastic means that certain events, e.g., the result of a weapon being fired or the impact of an artillery volley, occur according to laws of chance (random number generators and tables of probabilities of detection (PD), acquisition (PA), hit (PH), kill (PK), etc.). The principle modeling focus is on those military systems that participate in manuever and artillery operations. In addition to conventional direct fire and artillery operations, JANUS(T) models precision guided munitions, minefield employment and breaching, heat stress casualties, suppression, etc.

Throughout the development of JANUS(T), and its Janus precursor at Lawrence Livermore National Laboratory, extensive efforts have been made to make the model ‘‘user friendly.’’

C. QUALIFICATION PROCESS

Missing unit movements and initial force structures were filled in, often making ‘‘educated guesses’’ by combining information on the CIS tapes and the written portion of the take-home package.

This project effectively could not have proceeded if we had not been able to automate transfers of data between different databases and computer operating systems. CPT Mike Bowman, USA, wrote a thesis for LI [61], detailing his management of the many information-processing tasks associated with this project. He has coordinated and integrated data from Lawrence Livermore National Laboratory (LLNL), TRADOC (Training and Doctrine Command) Analysis Command (TRAC) at White Sands Missile Range (TRAC-WSMR) and at Monterey (TRAC-MTRY) for use by one of us (HF) for JANUS(T) wargaming at TRAC-MTRY, and for use by MW and LI for JANUS(T) and NTC modeling.

D. PRELIMINARY MATHEMATICAL MODELING OF NTC DATA

The ‘‘kills’’ attrition data from NTC and our JANUS(T) simulation at once looks strikingly similar during the force-on-force part of the combat. (See Fig. 4.)

Figure 4

From the single NTC trajectory qualified to date, 7 five-minute intervals in the middle of the battle were selected. From six JANUS(T) runs, similar force-on-force time epochs were identified, for a total of 42 data points. In the following fits, r represents Red tanks, and b represents Blue tanks.

Fitting NTC data to an additive noise model, a cost function of 2.08 gave:

$$\dot{r} = -2.49 \times 10^{-5} b - 4.97 \times 10^{-4} br + 0.320 \eta_r$$

$$\dot{b} = -2.28 \times 10^{-3} r - 3.23 \times 10^{-4} rb + 0.303 \eta_b \tag{30}$$

Fitting NTC data to a multiplicative noise model, a cost function of 2.16 gave:

$$\dot{r} = -5.69 \times 10^{-5} b - 4.70 \times 10^{-4} br + 1.06 \times 10^{-2} (1 + r) \eta_r$$

$$\dot{b} = -5.70 \times 10^{-4} r - 4.17 \times 10^{-4} rb + 1.73 \times 10^{-2} (1 + b) \eta_b \tag{31}$$

Fitting JANUS(T) data to an additive noise model, a cost function of 3.53 gave:

$$\dot{r} = -2.15 \times 10^{-5} b - 5.13 \times 10^{-4} br + 0.530 \eta_r$$

$$\dot{b} = -5.65 \times 10^{-3} r - 3.98 \times 10^{-4} rb + 0.784 \eta_b \tag{32}$$

Fitting JANUS(T) data to a multiplicative noise model, a cost function of 3.42 gave:

$$\dot{r} = -2.81 \times 10^{-4} b - 5.04 \times 10^{-4} br + 1.58 \times 10^{-2} (1 + r) \eta_r$$

$$\dot{b} = -3.90 \times 10^{-3} r - 5.04 \times 10^{-4} rb + 3.58 \times 10^{-2} (1 + b) \eta_b \tag{33}$$

This comparison illustrates that two different models about equally fit the short-time distribution. The multiplicative noise model shows that about a factor of 100 of the noise might be ‘‘divided out,’’ or understood in terms of the physical log-normal mechanism.

In order to discern which model best fits the data, we turn to the path-integral calculation of the long-time distribution, to see which model best follows the actual data.

Figs. 5 and 6 present the long-time probability of finding values of these forces. In general, the probability will be a highly nonlinear algebraic function, and there will be multiple peaks and valleys.

Figure 5

Figure 6

Figs. 7 and 8 give the means and variances of tank attrition from the JANUS(T) and NTC databases. Since we presently have only one NTC mission qualified, the variance of deviation from the mean is not really meaningful; it is given only to illustrate our approach which will be applied to more NTC missions as they are qualified and aggregated. Here, only the Blue JANUS(T) variances serve to distinguish the additive noise model as being consistent with the JANUS(T) data. Fig. 9 gives the exit probabilities of Blue and Red.

Figure 7

Figure 8

Figure 9

E. DISCUSSION OF STUDY

Data from 35 to 70 minutes was used for the short-time fit. The path integral used to calculate this fitted distribution from 35 minutes to beyond 70 minutes. This serves to compare long-time correlations in the mathematical model versus the data, and to help judge extrapolation past the data used for the short-time fits. It appears that indeed some multiplicative noise model is required. Of course, other Lanchester modelers most often do not consider noise at all, and at best just extract additive noise in the form of regression excesses. More work is required to find a better (or best?) algebraic form. The resulting form is required for input into higher echelon models. As more NTC data becomes available, we will also generate more JANUS(T) data. Then, we will be able to judge the best models with respect to how well they extrapolate across slightly different combat missions.

We have demonstrated proofs of principle, that battalion-level combat exercises can be well represented by the computer simulation JANUS(T), and that modern methods of nonlinear nonequilibrium statistical mechanics can well model these systems. Since only relatively simple drifts and diffusions were required, in larger systems, e.g., at brigade and division levels, it might be possible to ‘‘absorb’’ other important variables (C^3, human factors, logistics, etc.) into more nonlinear mathematical forms [62]. Otherwise, this battalion-level model should be supplemented with a ‘‘tree’’ of branches corresponding to estimated values of these variables.

F. EXTENSIONS TO OTHER SYSTEMS

CAPT Steve Upton, USMC, wrote a thesis for LI on the mathematical methodology [63]. Currently, he is looking at amphibious models, filling the gap in the spatial scales now using Air Force, Army, and Navy systems.

LT Jack Gallagher, USN, wrote a thesis for LI, documenting a Mideast Army-Navy joint scenario using a Battleship Battle Group with Tomahawk missiles supporting Air-Land combat [64]. LCDR Roy Balaconis, USN, wrote a thesis for LI, documenting the extension of this joint concept to a NATO scenario, including studies of Competitive Strategies and Integrated Strike Warfare, using two Carrier Battle Groups with Tomahawk and SLAM missiles, F-14 and A-6 tactical air support, and remotely piloted vehicles [65].

Issues of higher-echelon extrapolation. After fitting data from microscopic unit interactions to mesoscopic equations at battalion-regiment level, these equations can be used to drive higher level macroscopic scenarios at corps and theater levels. This mathematical aggregation is required for interpretation at multiple scales.

However, there are many issues yet to be resolved in using this approach. This requires approximately company-fidelity combat data from the unit interactions, e.g., which is barely the level obtained from NTC. It may be possible soon to obtain similar fidelity at division level, as NTC gears up for this scale of play.

Perhaps the biggest problem in using high-level aggregation is the representation of human factors. This is poorly represented in computer models. The premise is that perhaps some human factors are ‘‘absorbed’’ in the fitted SDE (stochastic differential equation) coefficients [62]. This premise must be tested, at least by extrapolating across battles. This is just another reason for the importance of including human factors whenever possible, e.g., having human wargamers. These problems are of course componded as we attempt wargames and simulations at larger scales: Synchronization of units at brigade level and management of brigades and battalions at division level are important aspects of real as well as simulated combat. For example, account must be taken of representation of combat service support and representation of IPB (intelligence, preparation of the battlefield).

Currently there are four approaches to modeling theater-level combat. (1) Distribution of combat scenarios: This approach in this paper uses stochastic trajectories of high-fidelity interactions and develops stochastic distributions of lower-echelon scenarios. Linear MOF's are derived and battle nodes are coordinated for theater combat. (2) Distribution of system-system interactions: This approach, e.g., in COSAGE used at CAA, uses statistical distributions of representative variables (including terrain and LOS) and distributions of system KVS's (Killer-Victim Scoreboards) to develop attrition model for theater KVS's. (3) Deterministic combat scenarios: This approach, e.g., used in VIC at TRAC-WSMR, develops KVS's from lower-echelon scenarios and uses system KVS's for theater models. (4) Theater stochastic high-fidelity model: This approach requires no aggregation, and studies all spatial-temporal scales simultaneously. This approach has regularly failed because of the huge computer resources required. Furthermore, aggregation is really required anyway, to simulate MOF's, MOE's, etc. required for decisions at various levels of command.

Basically, the important issues are: (1) sensitivity of theater models to different approaches, (2) inclusion/absorption of human factors into variables/parameters, (3) fidelity of representation of modern systems, e.g., cruise missiles, (possessing short reaction times, large spatial coverage, and C^3I at multiple scales), (4) statistical comparison of approaches, and (5)baselining of these approaches to some reality.

It seems clear that algebraic modeling with SDE is not sufficient to represent combat. I.e., there exist super-variables, e.g., especially for theater models. For example, to describe an operational procedure for modeling, specify levels of super-variables as: (I...) Level of combat, e.g., battalion-brigade; (A...) Terrain; (1...) Force structure; (a...) C^3; (i...) IPB. Then, perform multiple runs/trajectories for a given (I-A-1-a-i...). For example, some scenarios might degrade/jam communications between wargamers. Develop nonlinear stochastic multivariable (MOF's) mathematical models, specifying (I-A-1-a-i...) sets of runs, by performing short-time fits to data, fitting several possible math models, verifying these fits to long-time correlations, and then choosing the best math model. Then linearize for use in theater models, by finding multiple maxima of mathematical model probability distributions, and linearize these peaks at several engagement times.

To summarize, the overriding assumption in this approach is:

Dynamic Attrition = Faithful Measure of All Combat Variables

which must be supplemented by specifyinig the combat context in terms of Super-Variables.

Applications to process aggregated information. This statistical mechanics approach represents the mesoscale as a pattern-processing computer. The underlying mathematical theory, i.e., the path-integral approach, specifies a parallel-processing algorithm which statistically finds those parameter-regions of firing which contribute most to the overall probability distribution.

This is a kind of ‘‘intuitive’’ algorithm, globally searching a large multivariate data base to find parameter-regions deserving more detailed local information-processing. The derived probability distribution can be thought of as a filter, or processor, of incoming patterns of information; and this filter can be adapted, or updated, as it interacts with previously stored patterns of information.

CDR John Connell, USN [66], and LCDR Charles P. Yost, USN [67], have written theses for LI, examining multiple scales of interaction in large-scale systems, including combat systems.

These mathematical methods are quite general, and I have applied them to neuroscience, referenced here as the SMNI papers above, detailing properties of short-term memory derived from neuronal synaptic interactions, and calculating most likely frequencies observed in EEG data and velocities of propagation of information across neocortex. Working with Prof. Paul Nunez at Tulane University, we have detailed applications of this methodology to understand multiple scales of contributions to EEG data [38].

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FIGURE CAPTIONS

Fig. 1. Scaling C^3 systems

Fig. 2. Conventional Red versus Blue scenario

Fig. 3. Mathematical-model comparison process

Fig. 4. Attrition (‘‘kills’’) data is illustrated for an NTC mission and for three JANUS(T) runs using the NTC-qualified database.

Fig. 5. In this plot, the horizontal axes represent Red and Blue forces. For this JANUS(T) additive noise case, two time slices are superimposed. Taking the initial onset of the force-on-force part of the engagement as 35 minutes on the JANUS(T) clock, these peaks represent 50 and 100 minutes. Reflecting boundary conditions are taken at the beginning values of Red and Blue Tanks. Exit boundary conditions are taken at the other two surfaces.

Fig. 6. For the same case as in Fig. 5, contour plots are superimposed at 50, 70 and 100 minutes.

Fig. 7. JANUS(T) and NTC attrition means

Fig. 8. JANUS(T) and NTC attrition variances

Fig. 9. JANUS(T) and NTC attrition exit probabilities at 100 minutes for the same case as in Fig. 5

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